

Operator-Valued Measures, Dilations, and the Theory of Frames

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ABSTRACT. We develop elements of a general dilation theory for operator-valued measures. Hilbert space operator-valued measures are closely related to bounded linear maps on abelian von Neumann algebras, and some of our results include new dilation results for bounded linear maps that are not necessarily completely bounded, and from domain algebras that are not necessarily abelian. In the non-cb case the dilation space often needs to be a Banach space. We give applications to both the discrete and the continuous frame theory. There are natural associations between the theory of frames (including continuous frames and framings), the theory of operator-valued measures on sigma-algebras of sets, and the theory of continuous linear maps between C^* -algebras. In this connection frame theory itself is identified with the special case in which the domain algebra for the maps is an abelian von Neumann algebra and the map is normal (i.e. ultraweakly, or σ -weakly, or w^*) continuous. Some of the results for maps extend to the case where the domain algebra is non-commutative. It has been known for a long time that a necessary and sufficient condition for a bounded linear map from a unital C^* -algebra into $B(H)$ to have a Hilbert space dilation to a $*$ -homomorphism is that the mapping needs to be completely bounded. Our theory shows that even if it is not completely bounded it still has a Banach space dilation to a homomorphism. For the special case when the domain algebra is an abelian von Neumann algebra and the map is normal, we show that the dilation can be taken to be normal with respect to the usual Banach space version of ultraweak topology on the range space. We view these results as generalizations of the known result of Cazzaza, Han and Larson that arbitrary framings have Banach dilations, and also the known result that completely bounded maps have Hilbertian dilations. Our methods extend to some cases where the domain algebra need not be commutative, leading to new dilation results for maps of general von Neumann algebras. This paper was motivated by some recent results in frame theory and the observation that there is a close connection between the analysis of dual pairs of frames (both the discrete and the continuous theory) and the theory of operator-valued measures.

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Introduction

We investigate some natural associations between the theory of frames (including continuous frames and framings), the theory of operator-valued measures (OVM's) on sigma algebras of sets, and the theory of normal (ultraweakly or w^* continuous) linear maps on von Neumann algebras. Our main focus is on the dilation theory of these objects.

Generalized analysis-reconstruction schemes include dual pairs of frame sequences, framings, and continuous versions of these. We observe that all of these induce operator-valued measures on an appropriate σ -algebra of Borel sets in a natural way. The dilation theories for frames, dual pairs of frames, and framings, have been studied in the literature and many of their properties are well known. The continuous versions also have a dilation theory, but their properties are not as well understood. We show that all these can be perhaps better understood in terms of dilations of their operator-valued measures and their associated linear maps.

There is a well known dilation theory for those operator-valued measures that are completely bounded in the sense that their associated bounded linear maps between the operator algebra L^∞ of the sigma algebra and the algebra of all bounded linear operators on the underlying Hilbert space are completely bounded maps (cb maps for short). In this setting the dilation theory for operator-valued measures is obtained naturally from the dilation theory for cb maps, and cb maps dilate to $*$ -homomorphisms while OVM's dilate to projection-valued measures (PVM's), where the projections are orthogonal projections. We develop a general dilation theory for operator valued measures acting on Banach spaces where operator-valued measure (or maps) are not necessarily completely bounded. Our first main result (Theorem 2.31) shows that any operator-valued measure (not necessarily completely bounded) always has a dilation to a projection-valued measure acting on a Banach space. Here the dilation space often needs to be a Banach space and the projections are idempotents that are not necessarily self-adjoint (c.f. [DSc]).

Theorem A¹ *Let $E : \Sigma \rightarrow B(X, Y)$ be an operator-valued measure. Then there exist a Banach space Z , bounded linear operators $S : Z \rightarrow Y$ and $T : X \rightarrow Z$, and a projection-valued probability measure $F : \Sigma \rightarrow B(Z)$ such that*

$$E(B) = SF(B)T$$

for all $B \in \Sigma$.

¹We are enumerating what we feel are perhaps the most important of our contributions by labeling them A, B, C, D, E, ..., with the order not necessarily by order of importance but simply by the order of appearance in this manuscript. We thank the referee for making a suggestion along these lines in order to help the reader.

We will call (F, Z, S, T) in the above theorem a *Banach space dilation system*, and a *Hilbert dilation system* if Z can be taken as a Hilbert space. This theorem generalizes Naimark's (Neumark's) Dilation Theorem for positive operator valued measures. But even in the case that the underlying space is a Hilbert space the dilation space cannot always be taken to be a Hilbert space. Thus elements of the theory of Banach spaces are essential in this work. A key idea is the introduction of the elementary dilation space (Definitions 2.22 and 2.24) and the minimal dilation norm $\|\cdot\|_\alpha$ (Definition 2.28) on the space M_E of bounded measurable functions on the measure space for an OVM: The minimal dilation norm $\|\cdot\|_\alpha$ on M_E is defined by

$$\left\| \sum_{i=1}^N C_i E_{B_i, x_i} \right\|_\alpha = \sup_{B \in \Sigma} \left\| \sum_{i=1}^N C_i E(B \cap B_i) x_i \right\|_Y$$

for all $\sum_{i=1}^N C_i E_{B_i, x_i} \in M_E$. Using this we show that every OVM has a projection valued dilation to the elementary dilation space, and moreover, $\|\cdot\|_\alpha$ is a minimal norm on the elementary dilation space (see Theorem 2.26 and Theorem 2.30).

Theorem B *Let $E : \Sigma \rightarrow B(X, Y)$ be an operator-valued measure and (F, Z, S, T) be a corresponding Banach space dilation system. Then we have the following:*

(i) *There exist an elementary Banach space dilation system $(F_{\mathcal{D}}, \widetilde{M}_{E, \mathcal{D}}, S_{\mathcal{D}}, T_{\mathcal{D}})$ of E and a linear isometric embedding*

$$U : \widetilde{M}_{E, \mathcal{D}} \rightarrow Z$$

such that

$$S_{\mathcal{D}} = SU, \quad F(\Omega)T = UT_{\mathcal{D}}, \quad UF_{\mathcal{D}}(B) = F(B)U, \quad \forall B \in \Sigma.$$

(ii) *The norm $\|\cdot\|_\alpha$ is indeed a dilation norm. Moreover, If \mathcal{D} is a dilation norm of E , then there exists a constant $C_{\mathcal{D}}$ such that for any $\sum_{i=1}^N C_i E_{B_i, x_i} \in M_{E, \mathcal{D}}$,*

$$\sup_{B \in \Sigma} \left\| \sum_{i=1}^N C_i E(B \cap B_i) x_i \right\|_Y \leq C_{\mathcal{D}} \left\| \sum_{i=1}^N C_i E_{B_i, x_i} \right\|_{\mathcal{D}},$$

where $N > 0$, $\{C_i\}_{i=1}^N \subset \mathbb{C}$, $\{x_i\}_{i=1}^N \subset X$ and $\{B_i\}_{i=1}^N \subset \Sigma$. Consequently

$$\|f\|_\alpha \leq C_{\mathcal{D}} \|f\|_{\mathcal{D}}, \quad \forall f \in M_E.$$

Framings are the natural generalization of discrete frame theory (more specifically, dual-frame pairs) to non-Hilbertian settings. Even if the underlying space is a Hilbert space, the dilation space for framing induced operator valued measures can fail to be Hilbertian. This theory was originally developed by Casazza, Han and Larson in [CHL] as an attempt to introduce *frame theory with dilations* into a Banach space context. The initial motivation for the present manuscript was to completely understand the dilation theory of framings. In the context of Hilbert spaces, we realized that the dilation theory for discrete framings from [CHL] induces a dilation theory for discrete operator valued measures that may fail to be completely bounded in the sense of (c.f. [Pa]). While in general an operator-valued probability measure does not admit a Hilbert space dilation, the dilation theory can be strengthened in the case that it does admit a Hilbert space dilation (Theorem 3.3):

Theorem C *Let $E : \Sigma \rightarrow B(\mathcal{H})$ be an operator-valued probability measure. If E has a Hilbert dilation system $(\tilde{E}, \tilde{H}, S, T)$, then there exists a corresponding Hilbert dilation system (F, \mathcal{K}, V^*, V) such that $V : \mathcal{H} \rightarrow \mathcal{K}$ is an isometric embedding.*

This theorem turns out to have some interesting applications to framing induced operator valued measure dilation. In particular, it led to a complete characterization of framings whose induced operator valued measures are completely bounded. We include here a few sample examples with the following theorem:

Theorem D *Let $(x_i, y_i)_{i \in \mathbb{N}}$ be a non-zero framing for a Hilbert space \mathcal{H} , and E be the operator-valued probability measure induced by $(x_i, y_i)_{i \in \mathbb{N}}$. Then we have the following:*

(i) *E has a Hilbert dilation space \mathcal{K} if and only if there exist $\alpha_i, \beta_i \in \mathbb{C}, i \in \mathbb{N}$ with $\alpha_i \bar{\beta}_i = 1$ such that $\{\alpha_i x_i\}_{i \in \mathbb{N}}$ and $\{\beta_i y_i\}_{i \in \mathbb{N}}$ both are the frames for the Hilbert space \mathcal{H} .*

(ii) *E is a completely bounded map if and only if $\{x_i, y_i\}_{i \in \mathbb{N}}$ can be re-scaled to dual frames.*

(iii) *If $\inf \|x_i\| \cdot \|y_i\| > 0$, then we can find $\alpha_i, \beta_i \in \mathbb{C}, i \in \mathbb{N}$ with $\alpha_i \bar{\beta}_i = 1$ such that $\{\alpha_i x_i\}_{i \in \mathbb{N}}$ and $\{\beta_i y_i\}_{i \in \mathbb{N}}$ both are frames for the Hilbert space \mathcal{H} . Hence the operator-valued measure induced by $\{x_i, y_i\}_{i \in \mathbb{N}}$ has a Hilbertian dilation.*

For the existence of non-rescalable (to dual frame pairs) framings, we obtained the following:

Theorem E *There exists a framing for a Hilbert space such that its induced operator-valued measure is not completely bounded, and consequently it can not be re-scaled to obtain a framing that admits a Hilbert space dilation.*

The second part of Theorem E follows from the first part of the theorem and Theorem D (ii). This result also gives an example of a framing for a Hilbert space which is not rescalable to a dual frame pair. For the existence of such an example, the motivating example of framing constructed by Casazza, Han and Larson (Example 3.9 in [CHL]) can not be dilated to an unconditional basis for a Hilbert space, although it can be dilated to an unconditional basis for a Banach space (Theorem 4.6, [CHL]). We originally conjectured that this is an example that fails to induce a completely bounded operator valued measure. However, it turns out that this framing can be re-scaled to a framing that admits a Hilbert space dilation (see Theorem 5.4), and consequently disproves our conjecture. Our construction of the new example in Theorem E uses a non-completely bounded map to construct a non-completely bounded OVM which yields the required framing. This delimiting example shows that the dilation theory for framings developed in [CHL] gives a true generalization of Naimark's Dilation Theorem for the discrete case. This nontrivial example led us to consider general (non-necessarily-discrete) operator valued measures, and to the results of Chapter 2 that lead to the dilation theory for general (not necessarily completely bounded) OVM's that completely generalizes Naimark's Dilation theorem in a Banach space setting, and which is new even for Hilbert spaces.

Part (iii) of Theorem D provides us a sufficient condition under which a framing induced operator-valued measure has a Hilbert space dilation. This can be applied to framings that have nice structures. For example, the following is an unexpected

result for unitary system induced framings, where a unitary system is a countable collection of unitary operators. This clearly applies to wavelet and Gabor systems.

Corollary F *Let \mathcal{U}_1 and \mathcal{U}_2 be unitary systems on a separable Hilbert space \mathcal{H} . If there exist $x, y \in \mathcal{H}$ such that $\{\mathcal{U}_1 x, \mathcal{U}_2 y\}$ is a framing of \mathcal{H} , then $\{\mathcal{U}_1 x\}$ and $\{\mathcal{U}_2 y\}$ both are frames for \mathcal{H} .*

One of the important applications of our main dilation theorem (Theorem 2.31) is the dilation for not necessarily cb-maps with appropriate continuity properties from a commutative von Neumann algebra into $B(H)$ (Theorem 4.7):

Theorem G *If \mathcal{A} is a purely atomic abelian von Neumann algebra acting on a separable Hilbert space, then for every ultraweakly continuous linear map $\phi : \mathcal{A} \rightarrow B(\mathcal{H})$, there exists a Banach space Z , an ultraweakly continuous unital homomorphism $\pi : \mathcal{A} \rightarrow B(Z)$, and bounded linear operators $T : \mathcal{H} \rightarrow Z$ and $S : Z \rightarrow \mathcal{H}$ such that*

$$\phi(a) = S\pi(a)T$$

for all $a \in \mathcal{A}$.

The proof of Theorem G uses some special properties of the minimal dilation system for the ϕ induced operator valued measure on the space $(\mathbb{N}, 2^{\mathbb{N}})$. Motivated by some ideas used in the proof of Theorem G, we then obtained a universal dilation result, Theorem 4.10, for bounded linear mappings between Banach algebras.

Theorem H *Let \mathcal{A} be a Banach algebra, and let $\phi : \mathcal{A} \rightarrow B(\mathcal{H})$ be a bounded linear operator, where \mathcal{H} is a Banach space. Then there exists a Banach space Z , a bounded linear unital homomorphism $\pi : \mathcal{A} \rightarrow B(Z)$, and bounded linear operators $T : \mathcal{H} \rightarrow Z$ and $S : Z \rightarrow \mathcal{H}$ such that*

$$\phi(a) = S\pi(a)T$$

for all $a \in \mathcal{A}$.

We prove that this is a true generalization of our commutative theorem in an important special case (see Remark 4.11), and generalizes some of our results for maps of commutative von Neumann algebras to the case where the von Neumann algebra is non-commutative (see Theorem 4.12 and Corollary 4.13). For the case when \mathcal{A} is a von Neumann algebra acting on a separable Hilbert space and ϕ is ultraweakly continuous (i.e., normal) we conjecture that the dilation space Z can be taken to be separable and the dilation homomorphism π is also ultraweakly continuous. While we are not able to confirm this conjecture we shall prove the following:

Theorem I *Let K, H be Hilbert spaces, $A \subset B(K)$ be a von Neumann algebra, and $\phi : A \rightarrow B(H)$ be a bounded linear operator which is ultraweakly-SOT continuous on the unit ball B_A of A . Then there exists a Banach space Z , a bounded linear homeomorphism $\pi : A \rightarrow B(Z)$ which is SOT-SOT continuous on B_A , and bounded linear operator $T : H \rightarrow Z$ and $S : Z \rightarrow H$ such that*

$$\phi(a) = S\pi(a)T$$

for all $a \in A$. If in addition that K, H are separable, then the Banach space Z can be taken to be separable.

These results are apparently new for mappings of von Neumann algebras. They generalize special cases of Stinespring's Dilation Theorem. The standard discrete Hilbert space frame theory is identified with the special case of our theory in which the domain algebra is abelian and purely atomic, the map is completely bounded, and the OVM is purely atomic and completely bounded with rank-1 atoms (Remark 4.15).

The universal dilation result has connections with Kadison's similarity problem for bounded homomorphisms between von Neumann algebras (see the Remark 4.14). For example, if \mathcal{A} belongs to one of the following classes: nuclear; $\mathcal{A} = B(H)$; \mathcal{A} has no tracial states; \mathcal{A} is commutative; II_1 -factor with Murray and von Neumann's property Γ , then any non completely bounded map $\phi : \mathcal{A} \rightarrow B(H)$ can never have a Hilbertian dilation (i.e. the dilation space Z can never be a Hilbert space) since otherwise $\pi : \mathcal{A} \rightarrow B(Z)$ would be similar to a $*$ -homomorphism and hence completely bounded and so would be ϕ . On the other hand, if there exists a von Neumann algebra \mathcal{A} and a non completely bounded map ϕ from \mathcal{A} to $B(H)$ that has a Hilbert space dilation: $\pi : \mathcal{A} \rightarrow B(Z)$ (i.e., where Z is a Hilbert space), then π will be a counterexample to the Kadison's similarity problem since in this case π is a homomorphism that is not completely bounded and consequently can not be similar to a $*$ -homomorphism.

It is well known that there is a theory establishing a connection between general bounded linear mappings from the C^* -algebra $C(X)$ of continuous functions on a compact Hausdorff space X into $B(H)$ and operator valued measures on the sigma algebra of Borel subsets of X (c.f. [Pa]). If A is an abelian C^* -algebra then A can be identified with $C(X)$ for a topological space X and can also be identified with $C(\beta X)$ where βX is the Stone-Cech compactification of X . Then the support σ -algebra for the OVM is the sigma algebra of Borel subsets of βX which is enormous. However in our generalized (commutative) framing theory \mathcal{A} will always be an abelian von Neumann algebra presented up front as $L^\infty(\Omega, \Sigma, \mu)$, with Ω a topological space and Σ its algebra of Borel sets, and the maps on A into $B(H)$ are normal. In particular, to model the discrete frame and framing theory Ω is a countable index set with the discrete topology (most often \mathbb{N}), so Σ is its power set, and μ is counting measure. So in this setting it is more natural to work directly with this presentation in developing dilation theory rather than passing to $\beta\Omega$, and we take this approach in this paper.

We feel that the connection we make with established discrete frame and framing theory is transparent, and then the OVM dilation theory for the continuous case becomes a natural but nontrivial generalization of the theory for the discrete case that was inspired by framings. After doing this we attempted to apply our techniques to the case where the domain algebra for a map is non-commutative. This led to Theorem H. However, additional hypotheses are needed if dilations of maps are to have strong continuity and structural properties. For a map between C^* -algebras it is well-known that there is a Hilbert space dilation if the map is completely bounded. (If the domain algebra is commutative this statement is an iff.) Even if a map is not cb it has a Banach space dilation. We are interested in the continuity and structural properties a dilation can have. Theorem G shows that in the discrete abelian case, the dilation of a normal map can be taken to be normal and the dilation space can be taken to be separable, and Theorem I

tells us that with suitable hypotheses this type of result can be generalized to the noncommutative setting.

The dilation theory developed in this paper uses Hilbert space operator algebra theory and aspects of Banach space theory, so we try to present Banach space versions of Hilbert space results when we can obtain them. Some of the essential Hilbert space results we use are proven more naturally in a wider Banach context. The rest of the paper is organized as follows: Chapter one contains preliminary results and some exposition. In chapter two we develop our theory of Banach space operator-valued measures and the accompanying dilation theory. Operator valued measures have many different dilations to idempotent valued measures on larger Banach spaces (even if the measure to be dilated is a cb measure on a Hilbert space) and a part of the theory necessarily deals with classification issues. Chapter three is devoted to some additional results and exposition for Hilbert space operator-valued framings and measures, including the non-cb measures and their Banach dilations. It contains exposition and examples on the manner in which frames and framings on a Hilbert space induce natural operator valued measures on that Hilbert space. The reader might well benefit by reading this chapter first, although doing that would not be the natural order in which this theory is presented. In chapter four we present our results on dilations of linear maps in the non-commutative case. In chapter five, we give the detailed construction of the important example in Theorem E, and prove that the example constructed in [CHL] indeed induces a completely bounded operator valued measure.

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CHAPTER 1

Preliminaries

1.1. Frames

A *frame* \mathcal{F} for a Hilbert space \mathcal{H} is a sequence of vectors $\{x_n\} \subset \mathcal{H}$ indexed by a countable index set \mathbb{J} for which there exist constants $0 < A \leq B < \infty$ such that, for every $x \in \mathcal{H}$,

$$(1.1) \quad A\|x\|^2 \leq \sum_{n \in \mathbb{J}} |\langle x, x_n \rangle|^2 \leq B\|x\|^2.$$

The optimal constants (maximal for A and minimal for B) are known respectively as the upper and lower *frame bounds*. A frame is called a *tight frame* if $A = B$, and is called a *Parseval frame* if $A = B = 1$. If we only require that a sequence $\{x_n\}$ satisfies the upper bound condition in (1.1), then $\{x_n\}$ is also called a *Bessel sequence*.

A frame which is a basis is called a Riesz basis. Orthonormal bases are special cases of Parseval frames. It is elementary that a Parseval frame $\{x_n\}$ for a Hilbert space \mathcal{H} is an orthonormal basis if and only if each x_n is a unit vector.

For a Bessel sequence $\{x_n\}$, its *analysis operator* Θ is a bounded linear operator from \mathcal{H} to $\ell^2(\mathbb{N})$ defined by

$$(1.2) \quad \Theta x = \sum_{n \in \mathbb{N}} \langle x, x_n \rangle e_n,$$

where $\{e_n\}$ is the standard orthonormal basis for $\ell^2(\mathbb{N})$. It can be easily verified that

$$\Theta^* e_n = x_n, \quad \forall n \in \mathbb{N}$$

The Hilbert space adjoint Θ^* is called the *synthesis operator* for $\{x_n\}$. The positive operator $S := \Theta^* \Theta : \mathcal{H} \rightarrow \mathcal{H}$ is called the *frame operator*, or sometimes the *Bessel operator* if the Bessel sequence is not a frame, and we have

$$(1.3) \quad Sx = \sum_{n \in \mathbb{N}} \langle x, x_n \rangle x_n, \quad \forall x \in \mathcal{H}.$$

A sequence $\{x_n\}$ is a frame for \mathcal{H} if and only if its analysis operator Θ is bounded, injective and has closed range, which is, in turn, equivalent to the condition that the frame operator S is bounded and invertible. In particular, $\{x_n\}$ is a Parseval frame for \mathcal{H} if and only if Θ is an isometry or equivalently if $S = I$.

Let S be the frame operator for a frame $\{x_n\}$. Then the lower frame bound is $1/\|S^{-1}\|$ and the upper frame bound is $\|S\|$. From (1.3) we obtain the *reconstruction formula (or frame decomposition)*:

$$x = \sum_{n \in \mathbb{N}} \langle x, S^{-1} x_n \rangle x_n, \quad \forall x \in \mathcal{H}$$

or equivalently

$$x = \sum_{n \in \mathbb{N}} \langle x, x_n \rangle S^{-1} x_n, \quad \forall x \in \mathcal{H}.$$

(The second equation is obvious. The first can be obtained from the second by replacing x with $S^{-1}x$ and multiplying both sides by S .)

The frame $\{S^{-1}x_n\}$ is called the *canonical or standard dual* of $\{x_n\}$. In the case that $\{x_n\}$ is a Parseval frame for \mathcal{H} , we have that $S = I$ and hence

$$x = \sum_{n \in \mathbb{N}} \langle x, x_n \rangle x_n, \quad \forall x \in \mathcal{H}.$$

More generally, if a Bessel sequence $\{y_n\}$ satisfies

$$(1.4) \quad x = \sum_{n \in \mathbb{N}} \langle x, y_n \rangle x_n, \quad \forall x \in \mathcal{H},$$

where the convergence is in norm of \mathcal{H} , then $\{y_n\}$ is called an *alternate dual* of $\{x_n\}$. (Then $\{y_n\}$ is also necessarily a frame.) The canonical and alternate duals are usually simply referred to as *duals*, and $\{x_n, y_n\}$ is called a *dual frame pair*. It is a well-known fact that a frame $\{x_n\}$ is a Riesz basis if and only if $\{x_n\}$ has a unique dual frame (cf. [HL]).

There is a geometric interpretation of Parseval frames and general frames. Let P be an orthogonal projection from a Hilbert space \mathcal{K} onto a closed subspace \mathcal{H} , and let $\{u_n\}$ be a sequence in \mathcal{K} . Then $\{Pu_n\}$ is called the *orthogonal compression* of $\{u_n\}$ under P , and correspondingly $\{u_n\}$ is called an *orthogonal dilation* of $\{Pu_n\}$. We first observe that if $\{u_n\}$ is a frame for \mathcal{K} , then $\{Pu_n\}$ is a frame for \mathcal{H} with frame bounds at least as *good* as those of $\{u_n\}$ (in the sense that the lower frame cannot decrease and the upper bound cannot increase). In particular, $\{Pu_n\}$ is a Parseval frame for \mathcal{H} when $\{u_n\}$ is an orthonormal basis for \mathcal{K} , i.e., every orthogonal compression of an orthonormal basis (resp. Riesz basis) is a Parseval frame (resp. frame) for the projection subspace. The converse is also true: every frame can be orthogonally dilated to a Riesz basis, and every Parseval frame can be dilated to an orthonormal basis. This was apparently first shown explicitly by Han and Larson in Chapter 1 of [HL]. There, with appropriate definitions it had an elementary two-line proof. And as noted by several authors, it can be alternately derived by applying the Naimark (Neumark) Dilation theorem for operator valued measures by first passing from a frame sequence to a natural discrete positive operator-valued measure on the power set of the index set. So it is sometimes referred to as the Naimark dilation theorem for frames. In fact, this is the observation that inspired much of the work in this paper.

For completeness we formally state this result:

PROPOSITION 1.1. [HL] Let $\{x_n\}$ be a sequence in a Hilbert space \mathcal{H} . Then

- (i) $\{x_n\}$ is a Parseval frame for \mathcal{H} if and only if there exists a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and an orthonormal basis $\{u_n\}$ for \mathcal{K} such that $x_n = Pu_n$, where P is the orthogonal projection from \mathcal{K} onto \mathcal{H} .
- (ii) $\{x_n\}$ is a frame for \mathcal{H} if and only if there exists a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and a Riesz basis $\{v_n\}$ for \mathcal{K} such that $x_n = Pv_n$, where P again is the orthogonal projection from \mathcal{K} onto \mathcal{H} .

The above dilation result was later generalized in [CHL] to dual frame pairs.

THEOREM 1.2. *Suppose that $\{x_n\}$ and $\{y_n\}$ are two frames for a Hilbert space \mathcal{H} . Then the following are equivalent:*

- (i) $\{y_n\}$ is a dual for $\{x_n\}$;
- (ii) *There exists a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and a Riesz basis $\{u_n\}$ for \mathcal{K} such that $x_n = Pu_n$, and $y_n = Pu_n^*$, where $\{u_n^*\}$ is the (unique) dual of the Riesz basis $\{u_n\}$ and P is the orthogonal projection from \mathcal{K} onto \mathcal{H} .*

As in [CHL], a *framing* for a Banach space X is a pair of sequences $\{x_i, y_i\}$ with $\{x_i\}$ in X , $\{y_i\}$ in the dual space X^* of X , satisfying the condition that

$$x = \sum_i \langle x, y_i \rangle x_i,$$

where this series converges unconditionally for all $x \in X$.

The definition of a framing is a natural generalization of the definition of a dual frame pair. Assume that $\{x_i\}$ is a frame for \mathcal{H} and $\{y_i\}$ is a dual frame for $\{x_i\}$. Then $\{x_i, y_i\}$ is clearly a framing for \mathcal{H} . Moreover, if α_i is a sequence of non-zero constants, then $\{\alpha_i x_i, \bar{\alpha}_i^{-1} y_i\}$ (called a rescaling of the pair) is also a framing, although a simple example (Example 1.3) shows that it need not be a pair of frames, even if $\{\alpha_i x_i\}$, $\{\bar{\alpha}_i^{-1} y_i\}$ are bounded sequence.

EXAMPLE 1.3. Let \mathcal{H} be a separable Hilbert space and let $\{e_i\}_{i \in \mathbb{N}}$ be an orthonormal basis of \mathcal{H} . Let

$$\{x_i\} = \{e_1, e_2, e_2, e_3, e_3, e_3, \dots\}$$

and

$$\{y_i\} = \left\{ e_1, \frac{1}{2}e_2, \frac{1}{2}e_2, \frac{1}{3}e_3, \frac{1}{3}e_3, \frac{1}{3}e_3, \dots \right\}.$$

Then $\{x_i, y_i\}_{i \in \mathbb{N}}$ is a framing of \mathcal{H} , $\|x_i\| \leq 1$, $\|y_i\| \leq 1$, but neither $\{x_i\}$ nor $\{y_i\}$ are frames for \mathcal{H} .

PROOF. Let

$$\{\alpha_i\} = \left\{ 1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \dots \right\},$$

then

$$\{\alpha_i x_i\} = \{y_i / \alpha_i\} = \left\{ e_1, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \dots \right\}$$

is a Parseval frame of \mathcal{H} . Thus for any $x \in \mathcal{H}$, we have

$$x = \sum_{i \in \mathbb{N}} \langle x, \alpha_i x_i \rangle \frac{y_i}{\alpha_i} = \sum_{i \in \mathbb{N}} \langle x, x_i \rangle y_i,$$

and this series converges unconditionally. Hence $\{x_i, y_i\}_{i \in \mathbb{N}}$ is a framing of \mathcal{H} and $\|x_i\| \leq 1$, $\|y_i\| \leq 1$. But for any $j \in \mathbb{N}$,

$$\sum_{i \in \mathbb{N}} |\langle e_j, x_i \rangle|^2 = j \|e_j\|^2 = j$$

and

$$\sum_{i \in \mathbb{N}} |\langle e_j, y_i \rangle|^2 = \frac{\|e_j\|^2}{j} = \frac{1}{j}.$$

So $\{x_i\}$ is not a Bessel sequence and $\{y_i\}$ is a Bessel sequence but not a frame. \square

If $\{x_i, y_i\}$ is a framing, and if $\{x_i, y_i\}$ are both Bessel sequences, then $\{x_i\}, \{y_i\}$ are frames and $\{x_i, y_i\}$ is a dual frame pair. Indeed, if B, C are the Bessel bounds for $\{x_i\}, \{y_i\}$ respectively, then for each x ,

$$\begin{aligned} \langle x, x \rangle &= \left\langle \sum_{i \in \mathbb{N}} \langle x, y_i \rangle x_i, x \right\rangle \\ &= \sum_{i \in \mathbb{N}} \langle x, y_i \rangle \langle x_i, x \rangle \\ &\leq \left(\sum_{i \in \mathbb{N}} |\langle x, y_i \rangle|^2 \right)^{1/2} \left(\sum_{i \in \mathbb{N}} |\langle x_i, x \rangle|^2 \right)^{1/2} \\ &\leq C^{1/2} \|x\| \left(\sum_{i \in \mathbb{N}} |\langle x_i, x \rangle|^2 \right)^{1/2}, \end{aligned}$$

and so $C^{-1} \|x\|^2 \leq \sum_{i \in \mathbb{N}} |\langle x_i, x \rangle|^2$ as required. So our interests will involve framings for which $\{x_i\}, \{y_i\}$, or both, are not Bessel sequences.

DEFINITION 1.4. [CHL] A sequence $\{x_i\}_{i \in \mathbb{N}}$ in a Banach space X is a *projective frame* for X if there is a Banach space Z with an unconditional basis $\{z_i, z_i^*\}$ with $X \subset Z$ and a (onto) projection $P : Z \rightarrow X$ so that $Pz_i = x_i$ for all $i \in \mathbb{N}$. If $\{z_i\}$ is a 1-unconditional basis for Z and $\|P\| = 1$, we will call $\{x_i\}$ a Projective Parseval frame for X .

In this case, we have for all $x \in X$ that

$$x = \sum_i \langle x, z_i^* \rangle z_i = Px = \sum_i \langle x, z_i^* \rangle Pz_i = \sum_i \langle x, z_i^* \rangle x_i,$$

and this series converges unconditionally in X . So this definition recaptures the unconditional convergence from the Hilbert space definition.

We note that there exist projective frames in the sense of Definition 1.4 for an infinite dimensional Hilbert space that fail to be frames. An example is contained in Chapter 5.

DEFINITION 1.5. [CHL] A framing model is a Banach space Z with a fixed unconditional basis $\{e_i\}$ for Z . A framing modeled on $(Z, \{e_i\}_{i \in \mathbb{N}})$ for a Banach space X is a pair of sequences $\{y_i\}$ in X^* . and $\{x_i\}$ in X so that the operator $\theta : X \rightarrow Z$ defined by

$$\theta u = \sum_{i \in \mathbb{N}} \langle u, y_i \rangle e_i,$$

is an into isomorphism and $\Gamma : Z \rightarrow X$ given by

$$\Gamma \left(\sum_{i \in \mathbb{N}} a_i e_i \right) = \sum_{i \in \mathbb{N}} a_i x_i$$

is bounded and $\Gamma\theta = I_X$.

In this setting, Γ becomes the reconstruction operator for the frame. The following result due to Casazza, Han and Larson [CHL] shows that these three methods for defining a frame on a Banach space are really the same.

PROPOSITION 1.6. Let X be a Banach space and $\{x_i\}$ be a sequence of elements of X . The following are equivalent:

- (1) $\{x_i\}$ is a projective frame for X .
- (2) There exists a sequence $y_i \in X^*$ so that $\{x_i, y_i\}$ is a framing for X .
- (3) There exists a sequence $y_i \in X^*$ and a framing model $(Z, \{e_i\})$ so that $\{x_i, y_i\}$ is a framing modeled on $(Z, \{e_i\})$.

This proposition tells us that if $\{x_i, y_i\}$ is a framing of X , then $\{x_i, y_i\}$ can be dilated to an unconditional basis. That is, we can find a Banach space Z with an unconditional basis $\{e_i, e_i^*\}$, $X \subset Z$ and two bounded linear maps S and T such that $Se_i = x_i$ and $Te_i^* = y_i$.

DEFINITION 1.7. Let \mathcal{H} be a separable Hilbert space and Ω be a σ -locally compact (σ -compact and locally compact) Hausdorff space endowed with a positive Radon measure μ with $\text{supp}(\mu) = \Omega$. A weakly continuous function $\mathcal{F} : \Omega \rightarrow \mathcal{H}$ is called a *continuous frame* if there exist constants $0 < C_1 \leq C_2 < \infty$ such that

$$C_1 \|x\|^2 \leq \int_{\Omega} |\langle x, \mathcal{F}(\omega) \rangle|^2 d\mu(\omega) \leq C_2 \|x\|^2, \quad \forall x \in \mathcal{H}.$$

If $C_1 = C_2$ then the frame is called *tight*. Associated to \mathcal{F} is the frame operator $S_{\mathcal{F}}$ defined in the weak sense by

$$S_{\mathcal{F}} : \mathcal{H} \rightarrow \mathcal{H}, \quad \langle S_{\mathcal{F}}(x), y \rangle := \int_{\Omega} \langle x, \mathcal{F}(\omega) \rangle \cdot \langle \mathcal{F}(\omega), y \rangle d\mu(\omega).$$

It follows from the definition that $S_{\mathcal{F}}$ is a bounded, positive, and invertible operator. We define the following transform associated to \mathcal{F} ,

$$V_{\mathcal{F}} : \mathcal{H} \rightarrow L^2(\Omega, \mu), \quad V_{\mathcal{F}}(x)(\omega) := \langle x, \mathcal{F}(\omega) \rangle.$$

This operator is called the *analysis operator* in the literature and its adjoint operator is given by

$$V_{\mathcal{F}}^* : L^2(\Omega, \mu) \rightarrow \mathcal{H}, \quad \langle V_{\mathcal{F}}^*(f), x \rangle := \int_{\Omega} f(\omega) \langle \mathcal{F}(\omega), x \rangle d\mu(\omega).$$

Then we have $S_{\mathcal{F}} = V_{\mathcal{F}}^* V_{\mathcal{F}}$, and

$$(1.5) \quad \langle x, y \rangle = \int_{\Omega} \langle x, \mathcal{F}(\omega) \rangle \cdot \langle \mathcal{G}(\omega), y \rangle d\mu(\omega),$$

where $\mathcal{G}(\omega) := S_{\mathcal{F}}^{-1} \mathcal{F}(\omega)$ is the *standard dual* of \mathcal{F} . A weakly continuous function $\mathcal{F} : \Omega \rightarrow \mathcal{H}$ is called *Bessel* if there exists a positive constant C such that

$$\int_{\Omega} |\langle x, \mathcal{F}(\omega) \rangle|^2 d\mu(\omega) \leq C \|x\|^2, \quad \forall x \in \mathcal{H}.$$

It can be easily shown that if $\mathcal{F} : \Omega \rightarrow \mathcal{H}$ is Bessel, then it is a frame for \mathcal{H} if and only if there exists a Bessel mapping \mathcal{G} such that the reconstruction formula (1.5) holds. This \mathcal{G} may not be the standard dual of \mathcal{F} . We will call $(\mathcal{F}, \mathcal{G})$ a *dual pair*.

A discrete frame is a Riesz basis if and only if its analysis operator is surjective. But for a continuous frame \mathcal{F} , in general we don't have $V_{\mathcal{F}}(\mathcal{H}) = L^2(\Omega, \mu)$. In fact, this could happen only when μ is purely atomic. Therefore there is no Riesz basis type dilation theory for continuous frames (however, we will see later that in contrast the induced operator-valued measure does have projection valued measure dilations). The following modified dilation theorem was due to Gabardo and Han [GH]:

THEOREM 1.8. *Let \mathcal{F} be a (Ω, μ) -frame for \mathcal{H} and \mathcal{G} be one of its duals. Suppose that both $V_{\mathcal{F}}(\mathcal{H})$ and $V_{\mathcal{G}}(\mathcal{H})$ are contained in the range space \mathcal{M} of the analysis operator for some (Ω, μ) -frame. Then there is a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and a (Ω, μ) -frame $\tilde{\mathcal{F}}$ for \mathcal{K} with $P\tilde{\mathcal{F}} = \mathcal{F}$, $P\tilde{\mathcal{G}} = \mathcal{G}$ and $V_{\tilde{\mathcal{F}}}(\mathcal{H}) = \mathcal{M}$, where $\tilde{\mathcal{G}}$ is the standard dual of $\tilde{\mathcal{F}}$ and P is the orthogonal projection from \mathcal{K} onto \mathcal{H} .*

DEFINITION 1.9. Let X be a Banach space and Ω be a σ -locally compact Hausdorff space. Let μ be a Borel measure on ω . A continuous framing on X is a pair of maps $(\mathcal{F}, \mathcal{G})$,

$$\mathcal{F} : \Omega \rightarrow X, \quad \mathcal{G} : \Omega \rightarrow X^*,$$

such that the equation

$$\langle E_{(\mathcal{F}, \mathcal{G})}(B)x, y \rangle = \int_B \langle x, \mathcal{G}(\omega) \rangle \langle \mathcal{F}(\omega), y \rangle d\mu(\omega)$$

for $x \in X$, $y \in X^*$, and B a Borel subset of Ω , defines an operator-valued probability measure on Ω taking value in $B(X)$ (see Definition 2.1). In particular, we require the integral on the right to converge for each $B \subset \Omega$. We have

$$(1.6) \quad E_{(\mathcal{F}, \mathcal{G})}(B) = \int_B \mathcal{F}(\omega) \otimes \mathcal{G}(\omega) dE(\omega)$$

where the integral converges in the sense of Bochner. In particular, since $E_{(\mathcal{F}, \mathcal{G})}(\Omega) = I_X$, we have for any $x \in X$ that

$$\langle x, y \rangle = \int_{\Omega} \langle x, \mathcal{G}(\omega) \rangle \langle \mathcal{F}(\omega), y \rangle dE(\omega).$$

1.2. Operator-valued Measures

This section briefly discusses the well-known dilation theory for operator-valued measures, and establishes the connections between framing dilations and dilations of their associated operator-valued measures (more detailed discussion and investigation will be given in the subsequent chapters).

In operator theory, Naimark's dilation theorem is a result that characterizes positive operator-valued measures. Let Ω be a compact Hausdorff space, and let \mathcal{B} be the σ -algebra of all the Borel subsets of Ω . A $B(\mathcal{H})$ -valued measure on Ω is a mapping $E : \mathcal{B} \rightarrow B(\mathcal{H})$ that is weakly countably additive, i.e., if $\{B_i\}$ is a countable collection of disjoint Borel sets with union B , then

$$\langle E(B)x, y \rangle = \sum_i \langle E(B_i)x, y \rangle$$

holds for all x, y in \mathcal{H} . The measure is called *bounded* provided that

$$\sup\{\|E(B)\| : B \in \mathcal{B}\} < \infty,$$

and we let $\|\varphi\|$ denote this supremum. The measure is called *regular* if for all x, y in \mathcal{H} , the complex measure given by

$$(1.7) \quad \mu_{x,y}(B) = \langle E(B)x, y \rangle$$

is regular.

Given a regular bounded $B(\mathcal{H})$ -valued measure E , one obtains a bounded, linear map

$$\phi_E : C(\Omega) \rightarrow B(\mathcal{H})$$

by

$$(1.8) \quad \langle \phi_E(f)x, y \rangle = \int_{\Omega} f d\mu_{x,y}.$$

Conversely, given a bounded, linear map $\phi : C(\Omega) \rightarrow B(\mathcal{H})$, if one defines regular Borel measures $\{\mu_{x,y}\}$ for each x, y in \mathcal{H} by the above formula (1.8), then for each Borel set B , there exists a unique, bounded operator $E(B)$, defined by formula (1.7), and the map $B \rightarrow E(B)$ defines a bounded, regular $B(\mathcal{H})$ -valued measure. Thus, we see that there is a one-to-one correspondence between the bounded, linear maps of $C(\Omega)$ into $B(\mathcal{H})$ and the regular bounded $B(\mathcal{H})$ -valued measures. Such measures are called

- (i) *spectral* if $E(B_1 \cap B_2) = E(B_1) \cdot E(B_2)$,
- (ii) *positive* if $E(B) \geq 0$,
- (iii) *self-adjoint* if $E(B)^* = E(B)$,

for all Borel sets B, B_1 and B_2 .

Note that if E is spectral and self-adjoint, then $E(B)$ must be an orthogonal projection for all $B \in \mathcal{B}$, and hence E is positive.

THEOREM 1.10 (Naimark). *Let E be a regular, positive, $B(\mathcal{H})$ -valued measure on Ω . Then there exist a Hilbert space \mathcal{K} , a bounded linear operator $V : \mathcal{H} \rightarrow \mathcal{K}$, and a regular, self-adjoint, spectral, $B(\mathcal{K})$ -valued measure F on Ω , such that*

$$E(B) = V^* F(B) V.$$

Stinespring's dilation theorem is for completely positive maps on C^* -algebras. Let \mathcal{A} be a unital C^* -algebra. An operator-valued linear map $\phi : \mathcal{A} \rightarrow B(\mathcal{H})$ is said to be *positive* if $\phi(a^*a) \geq 0$ for every $a \in \mathcal{A}$, and it is called *completely positive* if for every n -tuple a_1, \dots, a_n of elements in \mathcal{A} , the matrix $(\phi(a_i^*a_j))$ is positive in the usual sense that for every n -tuple of vectors $\xi_1, \dots, \xi_n \in \mathcal{H}$, we have

$$(1.9) \quad \sum_{i,j=1}^n \langle \phi(a_i^*a_j) \xi_j, \xi_i \rangle \geq 0$$

or equivalently, $(\phi(a_i^*a_j))$ is a positive operator on the Hilbert space $\mathcal{H} \otimes \mathbb{C}^n$.

THEOREM 1.11 (Stinespring's dilation theorem). *Let \mathcal{A} be a unital C^* -algebra, and let $\phi : \mathcal{A} \rightarrow B(\mathcal{H})$ be a completely positive map. Then there exists a Hilbert space \mathcal{K} , a unital $*$ -homomorphism $\pi : \mathcal{A} \rightarrow B(\mathcal{K})$, and a bounded operator $V : \mathcal{H} \rightarrow \mathcal{K}$ with $\|\phi(1)\| = \|V\|^2$ such that*

$$\phi(a) = V^* \pi(a) V.$$

The following is also well known for commutative C^* -algebras:

THEOREM 1.12 (cf. Theorem 3.11, [Pa]). *Let \mathcal{B} be a C^* -algebra, and let $\phi : C(\Omega) \rightarrow \mathcal{B}$ be positive. Then ϕ is completely positive.*

This result together with Theorem 1.11 implies that Stinespring's dilation theorem holds for positive maps when \mathcal{A} is a unital commutative C^* -algebra.

A proof of Naimark dilation theorem by using Stinespring's dilation theorem can be sketched as follows: Let $\phi : \mathcal{A} \rightarrow B(\mathcal{H})$ be the natural extension of E to the C^* -algebra \mathcal{A} generated by all the characteristic functions of measurable subsets of Ω . Then ϕ is positive, and hence is completely positive by Theorem 1.12. Apply Stinespring's dilation theorem to obtain a $*$ -homomorphism $\pi : \mathcal{A} \rightarrow B(\mathcal{K})$, and a

bounded, linear operator $V : \mathcal{H} \rightarrow \mathcal{K}$ such that $\phi(f) = V^* \pi(f) V$ for all f in \mathcal{A} . Let F be the $B(\mathcal{K})$ -valued measure corresponding to π . Then it can be verified that F has the desired properties.

Let \mathcal{A} be a C^* -algebra. We use M_n to denote the set of all $n \times n$ complex matrices, and $M_n(\mathcal{A})$ to denote the set of all $n \times n$ matrices with entries from \mathcal{A} . For the following theory see (c.f. [Pa]):

Given two C^* -algebras \mathcal{A} and \mathcal{B} and a map $\phi : \mathcal{A} \rightarrow \mathcal{B}$, obtain maps $\phi_n : M_n(\mathcal{A}) \rightarrow M_n(\mathcal{B})$ via the formula

$$\phi_n((a_{i,j})) = (\phi(a_{i,j})).$$

The map ϕ is called completely bounded if ϕ is bounded and $\|\phi\|_{cb} = \sup_n \|\phi_n\|$ is finite.

Completely positive maps are completely bounded. In the other direction we have Wittstock's decomposition theorem [Pa]:

PROPOSITION 1.13. Let \mathcal{A} be a unital C^* -algebra, and let $\phi : \mathcal{A} \rightarrow B(\mathcal{H})$ be a completely bounded map. Then ϕ is a linear combination of two completely positive maps.

The following is a generalization of Stinespring's representation theorem.

THEOREM 1.14. Let \mathcal{A} be a unital C^* -algebra, and let $\phi : \mathcal{A} \rightarrow B(\mathcal{H})$ be a completely bounded map. Then there exists a Hilbert space \mathcal{K} , a $*$ -homomorphism $\pi : \mathcal{A} \rightarrow B(\mathcal{K})$, and bounded operators $V_i : \mathcal{H} \rightarrow \mathcal{K}, i = 1, 2$, with $\|\phi\|_{cb} = \|V_1\| \cdot \|V_2\|$ such that

$$\phi(a) = V_1^* \pi(a) V_2$$

for all $a \in \mathcal{A}$. Moreover, if $\|\phi\|_{cb} = 1$, then V_1 and V_2 may be taken to be isometries.

Now let Ω be a compact Hausdorff space, let E be a bounded, regular, operator-valued measure on Ω , and let $\phi : C(\Omega) \rightarrow B(\mathcal{H})$ be the bounded, linear map associated with E by integration as described in section 1.4.1. So for any $f \in C(\Omega)$,

$$\langle \phi(f)x, y \rangle = \int_{\Omega} f d\mu_{x,y},$$

where

$$\mu_{x,y}(B) = \langle E(B)x, y \rangle$$

The OVM E is called completely bounded when ϕ is completely bounded. Using Wittstock's decomposition theorem, E is completely bounded if and only if it can be expressed as a linear combination of positive operator-valued measures.

Let $\{x_i\}_{i \in \mathbb{J}}$ be a non-zero frame for a separable Hilbert space \mathcal{H} . Let Σ be the σ -algebra of all subsets of \mathbb{J} . Define the mapping

$$E : \Sigma \rightarrow B(\mathcal{H}), \quad E(B) = \sum_{i \in B} x_i \otimes x_i$$

where $x \otimes y$ is the mapping on \mathcal{H} defined by $(x \otimes y)(u) = \langle u, y \rangle x$. Then E is a regular, positive $B(\mathcal{H})$ -valued measure. By Naimark's dilation Theorem 1.10, there exists a Hilbert space \mathcal{K} , a bounded linear operator $V : \mathcal{H} \rightarrow \mathcal{K}$, and a regular, self-adjoint, spectral, $B(\mathcal{K})$ -valued measure F on \mathbb{J} , such that

$$E(B) = V^* F(B) V.$$

We will show (easily) that this Hilbert space \mathcal{K} can be ℓ_2 , and the atoms $x_i \otimes x_i$ of the measure dilates to rank-1 projections $e_i \otimes e_i$, where $\{e_i\}$ is the standard

orthonormal basis for ℓ^2 . That is \mathcal{K} can be the same as the dilation space in Proposition 1.1 (ii).

Similarly, suppose that $\{x_i, y_i\}_{i \in \mathbb{J}}$ is a non-zero framing for a separable Hilbert space \mathcal{H} . Define the mapping

$$E : \Sigma \rightarrow B(\mathcal{H}), \quad E(B) = \sum_{i \in B} x_i \otimes y_i,$$

for all $B \in \Sigma$. Then E is a $B(\mathcal{H})$ -valued measure. We will show that this E also has a dilation space Z . But this dilation space is not necessarily a Hilbert space, in general, it is a Banach space and consistent with Proposition 1.6. The dilation is essentially constructed using Proposition 1.6 (ii), where the dilation of the atoms $x_i \otimes y_i$ corresponds to the projection $u_i \otimes u_i^*$ and $\{u_i\}$ is an unconditional basis for the dilation space Z .

CHAPTER 2

Dilation of Operator-valued Measures

We develop a dilation theory of operator-valued measures to projection (i.e. idempotent)-valued measures. Our main results show that this can be always achieved for any operator-valued measure on a Banach space. Our approach is to dilate an operator-valued measure to a linear space which is “minimal” in a certain sense and complete it in a norm compatible with the dilation geometry. Such “dilation norms” are not unique, and we will focus on two: one we call the minimal dilation norm and one we call the maximal dilation norm. The applications of these new dilation results to Hilbert space operator-valued measures will be discussed in the subsequent chapters.

A positive operator-valued measure is a measure whose values are non-negative operators on a Hilbert space. From Naimark’s dilation Theorem, we know that every positive operator-valued measure can be dilated to a self-adjoint, spectral operator-valued measure on a larger Hilbert space. But not all of the operator-valued measures can have a Hilbert dilation space. For example, the operator-valued measure induced by the framing in Chapter 5 does not have a Hilbert dilation space. However it always admits a Banach dilation space, and moreover it can be dilated to a spectral operator-valued measure on a larger Banach space. Thus, it is necessary to develop the dilation theory to include *non-Hilbertian* operator-valued measures on a Hilbert space; i.e. $B(\mathcal{H})$ -valued measures that have no Hilbert dilation space. We obtain a similar result to Naimark’s dilation Theorem for non-Hilbertian $B(\mathcal{H})$ -valued measures. That is, for an arbitrary non-necessarily-Hilbertian $B(\mathcal{H})$ -valued measure E on (Ω, Σ) , we show that there exists a Banach space X , a spectral $B(X)$ -valued measure F on (Ω, Σ) , and bounded linear maps S and T such that $E(B) = SF(B)T$ holds for every $B \in \Sigma$.

2.1. Basic Definitions

DEFINITION 2.1. Let X and Y be Banach spaces, and let (Ω, Σ) be a measurable space. A $B(X, Y)$ -valued measure on Ω is a map $E : \Sigma \rightarrow B(X, Y)$ that is countably additive in the weak operator topology; that is, if $\{B_i\}$ is a disjoint countable collection of members of Σ with union B , then

$$y^*(E(B)x) = \sum_i y^*(E(B_i)x)$$

for all $x \in X$ and $y^* \in Y^*$.

We will use the symbol (Ω, Σ, E) if the range space is clear from context, or $(\Omega, \Sigma, E, B(X, Y))$, to denote this operator-valued measure system.

REMARK 2.2. The Orlicz-Pettis theorem states that weak unconditional convergence and norm unconditional convergence of a series are the same in every

Banach space (c.f.[DJT]). Thus we have that $\sum_i E(B_i)x$ weakly unconditionally converges to $E(B)x$ if and only if $\sum_i E(B_i)x$ strongly unconditionally converges to $E(B)x$. So Definition 2.1 is equivalent to saying that E is strongly countably additive, that is, if $\{B_i\}$ is a disjoint countable collection of members of Σ with union B , then

$$E(B)x = \sum_i E(B_i)x, \quad \forall x \in X.$$

DEFINITION 2.3. Let E be a $B(X, Y)$ -valued measure on (Ω, Σ) . Then the norm of E is defined by

$$\|E\| = \sup_{B \in \Sigma} \|E(B)\|.$$

We call E normalized if $\|E\| = 1$.

REMARK 2.4. A $B(X, Y)$ -valued measure E is always bounded, i.e.

$$(2.1) \quad \sup_{B \in \Sigma} \|E(B)\| < +\infty.$$

Indeed, for all $x \in X$ and $y^* \in Y^*$, $\mu_{x, y^*}(B) := y^*(E(B)x)$ is a complex measure on (Ω, Σ) . From complex measure theory (c.f.[Rud]), we know that μ_{x, y^*} is bounded, i.e.

$$\sup_{B \in \Sigma} |y^*(E(B)x)| < +\infty.$$

By the Uniform Boundedness Principle, we get (2.1).

DEFINITION 2.5. A $B(X)$ -valued measure E on (Ω, Σ) is called:

- (i) an operator-valued probability measure if $E(\Omega) = I_X$,
- (ii) a projection-valued measure if $E(B)$ is a projection on X for all $B \in \Sigma$,
- (iii) a spectral operator-valued measure if for all $A, B \in \Sigma$, $E(A \cap B) = E(A) \cdot E(B)$ (we will also use the term idempotent-valued measure to mean a spectral-valued measure.)

With Definition 2.1, a $B(X)$ -valued measure which is a projection-valued measure is always a spectral-valued measure (c.f. [Pa]). We give a proof for completeness. (Note that spectral operator-valued measures are clearly projection-valued measures).

LEMMA 2.6. Let E be a $B(X)$ -valued measure. If for each $B \in \Sigma$, $E(B)$ is a projection (i.e. idempotent), then for any $A, B \in \Sigma$, $E(A \cap B) = E(A) \cdot E(B)$.

PROOF. If P and Q are projections on X , then $P + Q$ is a projection on X if and only if $PQ = QP = 0$.

For disjoint $A, B \in \Sigma$, we have $E(A) + E(B) = E(A \cup B)$ which is a projection. From above, we obtain that $E(A) \cdot E(B) = E(A) \cdot E(B) = 0$. Then, for all $A, B \in \Sigma$, we have

$$\begin{aligned} E(A) \cdot E(B) &= E((A \cap B^c) \cup (A \cap B)) \cdot E((B \cap A^c) \cup (B \cap A)) \\ &= (E(A \cap B^c) + E(A \cap B)) \cdot (E(B \cap A^c) + E(B \cap A)) \\ &= E(A \cap B^c) \cdot E(B \cap A^c) + E(A \cap B^c) \cdot E(B \cap A) + \\ &\quad + E(A \cap B) \cdot E(B \cap A^c) + E(A \cap B) \cdot E(B \cap A) \\ &= E(A \cap B) \cdot E(B \cap A) = E(A \cap B). \end{aligned}$$

□

DEFINITION 2.7. Let $E : \Sigma \rightarrow B(X, Y)$ be an operator-valued measure. Then the dual of E , denoted by E^* , is a mapping from Σ to $B(Y^*, X^*)$ which is defined by $E^*(B) = [E(B)]^*$ for all $B \in \Sigma$.

PROPOSITION 2.8. Let X and Y be Banach spaces. If X is reflexive, then $E^* : \Sigma \rightarrow B(Y^*, X^*)$ is an operator-valued measure.

PROOF. Let $\{B_i\}$ be a disjoint countable collection of members of Σ with union B . Then for all $x \in X$ and $y^* \in Y^*$, we have

$$(E^*(B)y^*)(x) = y^*(E(B)x) = \sum_i y^*(E(B_i)x) = \sum_i (E^*(B_i)y^*)(x).$$

Since $X = X^{**}$, this shows that E is weakly countably additive in the sense of Definition 2.1. \square

REMARK 2.9. The following example shows that there exists an operator-valued measure E for which E^* is not weakly countably additive. So the above result is sharp.

EXAMPLE 2.10. Let $\{x_i, y_i\}_{i=1}^\infty \subset l_1 \times l_\infty$ and $\mathbf{1} = (1, 1, 1, \dots)$, where

$$\{x_i\}_{i=1}^\infty = \{e_1, e_1, e_1, e_2, e_3, e_4, e_5, e_6, \dots\}$$

$$\{y_i\}_{i=1}^\infty = \{\mathbf{1}, -\mathbf{1}, e_1^*, e_2^*, e_3^*, e_4^*, e_5^*, e_6^*, \dots\},$$

and $\{e_i\}_{i=1}^\infty$ is the standard unit vector basis. Let $\Omega = \mathbb{N}$, $\Sigma = 2^\mathbb{N}$ and

$$E(B) = \sum_{i \in B} x_i \otimes y_i.$$

Then it can be verified that E^* is not weakly countably additive.

Given two operator-valued measure space systems $(\Omega, \Sigma, E, B(X))$ and $(\Omega, \Sigma, F, B(Y))$.

We say that

- (i) E and F are *isometrically equivalent* (or *isometric*) if there is a surjective isometry $U : X \rightarrow Y$ such that $E(B) = U^{-1}F(B)U$ for all $B \in \Sigma$.
- (ii) E and F are *similar* (or *isomorphic*) if there is a bounded linear invertible operator $Q : X \rightarrow Y$ such that $E(B) = Q^{-1}F(B)Q$ for all $B \in \Sigma$.

The following property follows immediately from the definition:

PROPOSITION 2.11. Let X and Y be Banach spaces and let $E : \Sigma \rightarrow B(X, Y)$ be an operator-valued measure. Assume that W and Z are Banach spaces and that $T : W \rightarrow X$ and $S : Y \rightarrow Z$ both are bounded linear operators. Then the mapping $F : \Sigma \rightarrow B(W, Z)$ defined by

$$F(B) = SE(B)T, \quad \forall B \in \Sigma,$$

is an operator-valued measure.

Let X be a Banach space and $\{x_i, y_i\}_{i \in \mathbb{N}}$ be a framing for X . Then as we have introduced in Chapter 1 that the mapping E defined by

$$E : 2^\mathbb{N} \rightarrow B(X), \quad E(B) = \sum_{i \in B} x_i \otimes y_i,$$

is an operator-valued probability measure. We will call it the operator-valued probability measure *induced* by the framing $\{x_i, y_i\}_{i \in \mathbb{N}}$. The following lemma shows that every operator-valued probability measure system $(\mathbb{N}, 2^\mathbb{N}, E, B(X))$ with rank one atoms is induced by a framing.

LEMMA 2.12. *Let $(\mathbb{N}, 2^{\mathbb{N}}, E, B(X))$ be an operator-valued probability measure system with*

$$\text{rank}(E(\{i\})) \in \{0, 1\}$$

for all $i \in \mathbb{N}$. Then there exists a framing $\{x_i, y_i\}_{i \in \mathbb{N}}$ of X such that E is induced by $\{x_i, y_i\}_{i \in \mathbb{N}}$. Moreover, if X is a Hilbert space and E is spectral, then the framing $\{x_i, y_i\}_{i \in \mathbb{N}}$ can be chosen as a pair of Riesz bases of X .

PROOF. Without loss of generality, we can assume that for any $i \in \mathbb{N}$, $\text{rank}(E(\{i\})) = 1$. Then we can find $x_i \in X$ and $y_i \in X^*$ such that for any $x \in X$,

$$E(\{i\})(x) = y_i(x)x_i = (x_i \otimes y_i)(x).$$

It follows that

$$(2.2) \quad x = I_X(x) = E(\mathbb{N})(x) = \sum_{i \in \mathbb{N}} E(\{i\})(x) = \sum_{i \in \mathbb{N}} (x_i \otimes y_i)(x).$$

By Definition 2.1, we know that the series in (2.2) converges unconditionally for all $x \in X$. Hence $\{x_i, y_i\}_{i \in \mathbb{N}}$ is a framing of X .

When X is a Hilbert space and E is spectral, since for all $B \in 2^{\mathbb{N}}$,

$$E(B)x = \sum_{i \in B} \langle x, y_i \rangle x_i, \quad \forall x \in X,$$

we have for any $i, j \in \mathbb{N}$ and $x \in X$,

$$E(\{i\})E(\{j\})x = \langle x, y_j \rangle \langle x_j, y_i \rangle x_i.$$

When $i = j$, since

$$E(\{i\})E(\{i\})x = \langle x, y_i \rangle \langle x_i, y_i \rangle x_i = \langle x, y_i \rangle x_i = E(\{i\})x,$$

we get $\langle x_i, y_i \rangle = 1$.

When $i \neq j$, we have

$$E(\{i\})E(\{j\})x = \langle x, y_j \rangle \langle x_j, y_i \rangle x_i = 0,$$

and so $\langle x_j, y_i \rangle = 0$. If $\sum_i a_i x_i = 0$, then

$$E(\{j\}) \left(\sum_i a_i x_i \right) = \sum_i a_i E(\{j\})(x_i) = a_j x_j = 0,$$

and hence $a_j = 0$. Thus $\{x_i\}_{i \in \mathbb{N}}$ is an unconditional basis. By Lemma 3.6.2 in [Ch], we know that $\{x_i / \|x_i\|\}_{i \in \mathbb{N}}$ is a Riesz basis of X , and hence $\{\|x_i\| y_i\}_{i \in \mathbb{N}}$ is also a Riesz basis of X . Clearly for all $B \in 2^{\mathbb{N}}$, we also have

$$E(B)x = \sum_{i \in B} \langle x, \|x_i\| y_i \rangle x_i / \|x_i\|, \quad \forall x \in X.$$

□

2.2. Dilation Spaces and Dilations

Let (Ω, Σ) be a measurable space and $E : \Sigma \rightarrow B(X, Y)$ be an operator-valued measure.

DEFINITION 2.13. A Banach space Z is called a **dilation space** of an operator-valued measure space (Ω, Σ, E) if there exist bounded linear operators $S : Z \rightarrow Y$ and $T : X \rightarrow Z$, and a projection-valued measure space $(\Omega, \Sigma, F, B(Z))$ such that for any $B \in \Sigma$,

$$E(B) = SF(B)T.$$

We call S and T the corresponding analysis operator and synthesis operator, respectively, and use $(\Omega, \Sigma, F, B(Z), S, T)$ to denote the corresponding dilation projection-valued measure space system.

It is easy to see that the mappings

$$G_1 : \Sigma \rightarrow B(X, Z), \quad G_1(B) = F(B)T$$

and

$$G_2 : \Sigma \rightarrow B(X, Z), \quad G_2(B) = SF(B)$$

are both operator-valued measures. We call G_1 and G_2 the corresponding *analysis operator-valued measure* and *synthesis operator-valued measure*, respectively.

Clearly, $E = G_2 \cdot G_1$. Indeed,

$$E(B) = SF(B)T = SF(B)F(B)T = G_2(B)G_1(B).$$

REMARK 2.14. If $X = Y$ and $E(\Omega) = I_X$, then S is a surjection, T is an isomorphic embedding, and $TS : Z \rightarrow Z$ is a projection onto $T(X)$.

Further, we have the following result:

LEMMA 2.15. Let (Ω, Σ, E) be an operator-valued measure space and $(\Omega, \Sigma, F, B(Z), S, T)$ be a corresponding dilation projection-valued measure space system.

- (i) If $E(\Omega) : X \rightarrow Y$ is an isomorphic operator, then $S : Z \rightarrow Y$ is a surjection and $T : X \rightarrow Z$ is an isomorphic embedding.
- (ii) If $X = Y$ and $E(\Omega) = I_X$, then $TSF(\Omega)$, $F(\Omega)TS$ and $F(\Omega)TSF(\Omega)$ are all projections on Z .

PROOF. (i) Since $E(\Omega)$ is an isomorphic operator, for any $y \in Y$, there exists $x \in X$ such that $E(\Omega)x = y$. Hence $(SF(\Omega)T)(x) = E(\Omega)(x) = y$, and so S is surjective.

Suppose that $Tx_1 = Tx_2$ for $x_1, x_2 \in X$. Then we have $(SF(\Omega)T)(x_1) = (SF(\Omega)T)(x_2)$, and so $E(\Omega)x_1 = E(\Omega)x_2$. Since $E(\Omega)$ is invertible, we get $x_1 = x_2$. Thus T is an isomorphic embedding.

(ii) When $X = Y$ and $E(\Omega) = I_X$, we have

$$\begin{aligned} (TSF(\Omega))^2 &= TSF(\Omega)TSF(\Omega) = TE(\Omega)SF(\Omega) \\ &= TI_XSF(\Omega) = TSF(\Omega), \\ (F(\Omega)TS)^2 &= F(\Omega)TSF(\Omega)TS = F(\Omega)TI_XS \\ &= F(\Omega)TS, \end{aligned}$$

and

$$\begin{aligned}
(F(\Omega)TSF(\Omega))^2 &= F(\Omega)TSF(\Omega)F(\Omega)TSF(\Omega) \\
&= F(\Omega)TSF(\Omega)TSF(\Omega) \\
&= F(\Omega)TI_XSF(\Omega) \\
&= F(\Omega)TSF(\Omega).
\end{aligned}$$

Thus $TSF(\Omega)$, $F(\Omega)TS$ and $F(\Omega)TSF(\Omega)$ are all projections on Z . \square

In general, the corresponding dilation projection-valued measure system is not unique. However, we can always find a projection-valued probability measure system from a known dilation projection-valued measure system.

PROPOSITION 2.16. Let (Ω, Σ, E) be an operator-valued measure space and let $(\Omega, \Sigma, F, B(Z), S, T)$ be a corresponding dilation projection-valued measure space system. Define a mapping $\hat{F} : \Sigma \rightarrow B(F(\Omega)Z)$ by

$$\hat{F}(B) = F(B)|_{F(\Omega)Z}, \quad \text{for all } B \in \Sigma.$$

Then

$$\hat{S} = S|_{F(\Omega)Z} : F(\Omega)Z \rightarrow Y$$

and

$$\hat{T} = F(\Omega)T : X \rightarrow F(\Omega)Z$$

are both bounded linear operators. And \hat{F} is a spectral operator-valued probability measure such that

$$E(B) = \hat{S}\hat{F}(B)\hat{T}, \quad B \in \Sigma.$$

PROOF. Clearly \hat{S} and \hat{T} are both well defined and are bounded linear operators. Since

$$F(B)F(\Omega)Z = F(B)Z = F(\Omega)F(B)Z \subset F(\Omega)Z,$$

\hat{F} is well defined. It is clearly that \hat{F} is an operator-valued probability measure. Moreover, for all $A, B \in \Sigma$,

$$\begin{aligned}
\hat{F}(A)\hat{F}(B) &= F(A)|_{F(\Omega)Z}F(B)|_{F(\Omega)Z} \\
&= F(A)F(B)|_{F(\Omega)Z} \\
&= F(A \cap B)|_{F(\Omega)Z} \\
&= \hat{F}(A \cap B).
\end{aligned}$$

It follows that \hat{F} is a spectral operator-valued measure. Finally, we have for any $B \in \Sigma$ and $x \in X$,

$$\begin{aligned}
\hat{S}\hat{F}(B)\hat{T}(x) &= S|_{F(\Omega)Z}F(B)|_{F(\Omega)Z}F(\Omega)T(x) \\
&= S|_{F(\Omega)Z}F(B)F(\Omega)T(x) \\
&= SF(B)T(x) \\
&= E(B)(x).
\end{aligned}$$

\square

REMARK 2.17. It is easy to see that

$$\hat{S}\hat{T} = \hat{S}\hat{F}(\Omega)\hat{T} = E(\Omega).$$

If $X = Y$ and $E(\Omega) = I_X$, then

$$\hat{T}\hat{S} = \hat{T}SF(\Omega) = F(\Omega)T\hat{S} = F(\Omega)TSF(\Omega)$$

is a projection from $F(\Omega)Z$ onto $\hat{T}Z = F(\Omega)T(Z)$.

As a consequence of Proposition 2.16, we obtain:

COROLLARY 2.18. If an operator-valued measure can be dilated to a spectral operator-valued measure, then it can be dilated to a spectral operator-valued probability measure.

2.3. Elementary Dilation Spaces

This section provides the first step in the construction of dilation spaces for operator-valued measures. The main result (Theorem 2.26) shows that for any dilation projection-valued measure system $(\Omega, \Sigma, F, B(Z), S, T)$ of an operator-valued measure system $(\Omega, \Sigma, E, B(X, Y))$, there exists an “elementary” dilation operator-valued measure system which can be linearly isometrically embedded into $(\Omega, \Sigma, F, B(Z), S, T)$.

DEFINITION 2.19. Let Y be a Banach space and (Ω, Σ) be a measurable space. A mapping $\nu : \Sigma \rightarrow Y$ is called a **vector-valued measure** if ν is countably additive; that is, if $\{B_i\}$ is a disjoint countable collection of members of Σ with union B , then

$$\nu(B) = \sum_i \nu(B_i).$$

We use the notation (Ω, Σ, ν, Y) for a vector-valued measure system.

REMARK 2.20. By the Orlicz-Pettis Theorem, we know that ν countably additive is equivalent to ν weakly countably additive. That is

$$y^*(\nu(B)) = \sum_i y^*(\nu(B_i)),$$

for all $y^* \in Y^*$.

We use the symbol \mathfrak{M}_Σ^Y to denote the linear space of all vector-valued measures on (Ω, Σ) of Y .

Let X, Y be Banach spaces and $(\Omega, \Sigma, E, B(X, Y))$ an operator-valued measure system. For any $B \in \Sigma$ and $x \in X$, define

$$E_{B,x} : \Sigma \rightarrow Y, \quad E_{B,x}(A) = E(B \cap A)x, \quad \forall A \in \Sigma.$$

Then it is easy to see that $E_{B,x}$ is a vector-valued measure on (Ω, Σ) of Y and $E_{B,x} \in \mathfrak{M}_\Sigma^Y$.

Let $M_E = \text{span}\{E_{B,x} : x \in X, B \in \Sigma\}$. Obviously, $M_E \subset \mathfrak{M}_\Sigma^Y$, and we will refer it as the space induced by $(\Omega, \Sigma, E, B(X, Y))$. We first introduce some linear mappings on the spaces X, Y and M_E .

LEMMA 2.21. Let X, Y be Banach spaces and $(\Omega, \Sigma, E, B(X, Y))$ an operator-valued measure system. For any $\{C_i\}_{i=1}^N \subset \mathbb{C}$, $\{B_i\}_{i=1}^N \subset \Sigma$ and $\{x_i\}_{i=1}^N \subset X$, the mappings

$$\begin{aligned} S : M_E &\rightarrow Y, & S\left(\sum_{i=1}^N C_i E_{B_i, x_i}\right) &= \sum_{i=1}^N C_i E(B_i)x_i \\ T : X &\rightarrow M_E, & T(x) &= E_{\Omega, x} \end{aligned}$$

and

$$F(B) : M_E \rightarrow M_E, \quad F(B) \left(\sum_{i=1}^N C_i E_{B_i, x_i} \right) = \sum_{i=1}^N C_i E_{B \cap B_i, x_i}, \quad \forall B \in \Sigma$$

are well-defined and linear.

PROOF. Obviously T is well-defined. For any $\{C_i\}_{i=1}^N \subset \mathbb{C}$, $\{x_i\}_{i=1}^N \subset X$ and $\{B_i\}_{i=1}^N \subset \Sigma$, if

$$\sum_{i=1}^N C_i E_{B_i, x_i} = 0,$$

then, by definition of E_{x_i, B_i} , we have for any $A \in \Sigma$ that

$$\sum_{i=1}^N C_i E_{B_i, x_i}(A) = \sum_{i=1}^N C_i E(B_i \cap A) x_i = 0.$$

Let $B = \Omega$. Then

$$\sum_{i=1}^N C_i E(B_i) x_i = 0.$$

Hence S is well-defined. Similarly we can verify that $F(B)$ is also well-defined. Clearly for any $B \in \Sigma$, S, T and $F(B)$ are linear. \square

With the aid of Lemma 2.21, we can now give the definition of a dilation norm.

Note: In this manuscript we will in most cases use the traditional notation $\|\cdot\|$ for a norm; however, in the case of dilation norms (especially) we will frequently find it convenient to use the functional notation, typically $\mathcal{D}(\cdot)$, for a norm, because of the length of the expressions being normed. In this case we will sometimes also write $\|\cdot\|_{\mathcal{D}}$ for this same norm when the meaning is clear, using the norming function \mathcal{D} to subscript the traditional norm notation.

DEFINITION 2.22. Let M_E be the space induced by $(\Omega, \Sigma, E, B(X, Y))$. Let $\|\cdot\|$ be a norm on M_E . Denote this normed space by $M_{E, \|\cdot\|}$ and its completion $\widetilde{M}_{E, \|\cdot\|}$. The norm on $\widetilde{M}_{E, \|\cdot\|}$, with $\|\cdot\| := \|\cdot\|_{\mathcal{D}}$ given by a norming function \mathcal{D} as discussed above, is called a **dilation norm of E** if the following conditions are satisfied:

- (i) The mapping $S_{\mathcal{D}} : \widetilde{M}_{E, \mathcal{D}} \rightarrow Y$ defined on M_E by

$$S_{\mathcal{D}} \left(\sum_{i=1}^N C_i E_{B_i, x_i} \right) = \sum_{i=1}^N C_i E(B_i) x_i$$

is bounded.

- (ii) The mapping $T_{\mathcal{D}} : X \rightarrow \widetilde{M}_{E, \mathcal{D}}$ defined by

$$T_{\mathcal{D}}(x) = E_{\Omega, x}$$

is bounded.

- (iii) The mapping $F_{\mathcal{D}} : \Sigma \rightarrow B(\widetilde{M}_{E, \mathcal{D}})$ defined by

$$F_{\mathcal{D}}(B) \left(\sum_{i=1}^N C_i E_{B_i, x_i} \right) = \sum_{i=1}^N C_i E_{B \cap B_i, x_i}$$

is an operator-valued measure,

where $\{C_i\}_{i=1}^N \subset \mathbb{C}$, $\{x_i\}_{i=1}^N \subset X$ and $\{B_i\}_{i=1}^N \subset \Sigma$.

THEOREM 2.23. *Let $(\Omega, \Sigma, E, B(X, Y))$ be an operator-valued measure system. If a norm \mathcal{D} is a dilation norm of E , then the Banach space $\widetilde{M}_{E, \mathcal{D}}$ is a dilation space of $(\Omega, \Sigma, E, B(X, Y))$. Moreover, $(\Omega, \Sigma, F_{\mathcal{D}}, B(\widetilde{M}_{E, \mathcal{D}}), S_{\mathcal{D}}, T_{\mathcal{D}})$ is the corresponding dilation projection-valued probability measure system.*

PROOF. For any $x \in X$ and $B_1, B_2, A \in \Sigma$, we have

$$F_{\mathcal{D}}(B_1 \cap B_2)(E_{A,x}) = E_{B_1 \cap B_2 \cap A, x} = F_{\mathcal{D}}(B_1)(E_{B_2 \cap A, x}) = F_{\mathcal{D}}(B_1)F_{\mathcal{D}}(B_2)(E_{A,x}).$$

So $F_{\mathcal{D}}$ is a spectral operator-valued measure.

For any $B \in \Sigma$, we get

$$S_{\mathcal{D}}F_{\mathcal{D}}(B)T_{\mathcal{D}}(x) = S_{\mathcal{D}}F_{\mathcal{D}}(B)(E_{\Omega, x}) = S_{\mathcal{D}}(E_{B,x}) = E(B)(x), \quad \forall x \in X,$$

and thus $S_{\mathcal{D}}F_{\mathcal{D}}(E)T_{\mathcal{D}} = E(B)$. Therefore $(\Omega, \Sigma, F_{\mathcal{D}}, B(\widetilde{M}_{\varphi, \mathcal{D}}), S_{\mathcal{D}}, T_{\mathcal{D}})$ is the corresponding dilation projection-valued measure system.

Observe that for any $x \in X$ and $B \in \Sigma$, we have $F_{\mathcal{D}}(\Omega)(E_{B,x}) = E_{B,x}$. Hence $F_{\mathcal{D}}(\Omega) = I_{\widetilde{M}_{\varphi, \mathcal{D}}}$. The proof is complete. \square

DEFINITION 2.24. Let M_E be the space induced by $(\Omega, \Sigma, E, B(X, Y))$ and \mathcal{D} be a dilation norm of E . We call the Banach space $\widetilde{M}_{E, \mathcal{D}}$ **the elementary dilation space of E** and $(\Omega, \Sigma, F_{\mathcal{D}}, B(\widetilde{M}_{E, \mathcal{D}}), S_{\mathcal{D}}, T_{\mathcal{D}})$ the elementary dilation operator-valued measure system.

In order to prove the main result (the existence of a dilation norm) in this section, we need the following lemma:

LEMMA 2.25. *Let X and Y be Banach spaces, and let (Ω, Σ) be a measurable space. Assume that $X_0 \subset X$ is dense in X . If a mapping $E : \Sigma \rightarrow B(X, Y)$ is strongly countably additive on X_0 , i.e. if $\{B_i\}$ is a disjoint countable collection of members of Σ with union B then*

$$(2.3) \quad E(B)x = \sum_i E(B_i)x, \quad \forall x \in X_0,$$

and if E is uniformly bounded on X_0 , i.e. there exists a constant $C > 0$ such that for any $B \in \Sigma$

$$\|E(B)x\| \leq C\|x\|, \quad \forall x \in X_0,$$

then E is an operator-valued measure.

PROOF. Since $\overline{X_0} = X$, E is uniformly bounded on X . For any $N > 0$, by (2.3), if $\{B_i\}_{i=1}^N$ is a disjoint collection of members of Σ , then

$$(2.4) \quad E\left(\bigcup_{i=1}^N B_i\right)x = \sum_{i=1}^N E(B_i)x, \quad \forall x \in X_0.$$

Hence $E\left(\bigcup_{i=1}^N B_i\right) = \sum_{i=1}^N E(B_i)$ on X .

We have

$$\left\|\sum_{i=1}^N E(B_i)\right\| = \left\|E\left(\bigcup_{i=1}^N B_i\right)\right\| \leq C.$$

Let $\{B_i\}_{i=1}^\infty$ be a countable disjoint collection of elements of Σ with union B . For any $x \in X$, there exists a sequence $\{x_j\}_{j=1}^\infty \subset X_0$ such that

$$\lim_{j \rightarrow \infty} x_j = x.$$

Observe that

$$\begin{aligned}
& \left\| E(B)x - \sum_{i=1}^N E(B_i)x \right\| \\
&= \left\| E(B)x - E(B)x_M + E(B)x_M - \sum_{i=1}^N E(B_i)x_M + \sum_{i=1}^N E(B_i)x_M - \sum_{i=1}^N E(B_i)x \right\| \\
&\leq \|E(B)x - E(B)x_M\| + \left\| E(B)x_M - \sum_{i=1}^N E(B_i)x_M \right\| + \left\| \sum_{i=1}^N E(B_i)x_M - \sum_{i=1}^N E(B_i)x \right\| \\
&\leq \|E(B)\| \cdot \|x - x_M\| + \left\| E(B)x_M - \sum_{i=1}^N E(B_i)x_M \right\| + \left\| E\left(\bigcup_{i=1}^N B_i\right) \right\| \cdot \|x_M - x\| \\
&\leq 2C \|x - x_M\| + \left\| E(B)x_M - \sum_{i=1}^N E(B_i)x_M \right\|.
\end{aligned}$$

For any $\varepsilon > 0$, we can find $M > 0$ such that

$$\|x - x_M\| \leq \frac{\varepsilon}{3C}.$$

By (2.3), for $x_M \in X_0$, we can find a sufficiently large $N > 0$ such that

$$\left\| E(B)x_M - \sum_{i=1}^N E(B_i)x_M \right\| \leq \frac{\varepsilon}{3}.$$

Therefore

$$\left\| E(B)x - \sum_{i=1}^N E(B_i)x \right\| \leq \varepsilon$$

when N is sufficiently large. Hence

$$E(B)x = \sum_i E(B_i)x, \quad \forall x \in X.$$

Therefore E is an operator-valued measure. \square

Now we state and prove the main result of this section.

THEOREM 2.26. *Let $(\Omega, \Sigma, E, B(X, Y))$ be an operator-valued measure system and $(\Omega, \Sigma, F, B(Z), S, T)$ be a corresponding dilation projection-valued measure space system. Then there exist an elementary dilation operator-valued measure system $(\Omega, \Sigma, F_{\mathcal{D}}, B(\widetilde{M}_{E, \mathcal{D}}), S_{\mathcal{D}}, T_{\mathcal{D}})$ of E and a linear isometric embedding*

$$U : \widetilde{M}_{E, \mathcal{D}} \rightarrow Z$$

such that

$$S_{\mathcal{D}} = SU, \quad F(\Omega)T = UT_{\mathcal{D}}, \quad UF_{\mathcal{D}}(B) = F(B)U, \quad \forall B \in \Sigma.$$

PROOF. Define $\mathcal{D} : M_E \rightarrow R^+ \cup \{0\}$ by

$$\mathcal{D} \left(\sum_{i=1}^N C_i E_{B_i, x_i} \right) = \left\| \sum_{i=1}^N C_i F(B_i) T(x_i) \right\|_Z,$$

where $N > 0$, C_i are all complex numbers and $\{x_i\}_{i=1}^N \subset X$ and $\{B_i\}_{i=1}^N \subset \Sigma$. We first show that \mathcal{D} is a norm on M_E .

(i) Obviously,

$$\mathcal{D} \left(\sum_{i=1}^N C_i E_{B_i, x_i} \right) = \left\| \sum_{i=1}^N C_i F(B_i) T(x_i) \right\|_Z \geq 0.$$

(ii) Take $\{\tilde{C}_j\}_{j=1}^M, \{y_j\}_{j=1}^M \subset X$ and $\{A_j\}_{j=1}^M \subset \Sigma$. We have

$$\begin{aligned} \mathcal{D} \left(\sum_{i=1}^N C_i E_{B_i, x_i} + \sum_{j=1}^M \tilde{C}_j E_{A_j, y_j} \right) &= \left\| \sum_{i=1}^N C_i F(B_i) T(x_i) + \sum_{j=1}^M \tilde{C}_j F(A_j) T(y_j) \right\|_Z \\ &\leq \left\| \sum_{i=1}^N C_i F(B_i) T(x_i) \right\|_Z + \left\| \sum_{j=1}^M \tilde{C}_j F(A_j) T(y_j) \right\|_Z \\ &= \mathcal{D} \left(\sum_{i=1}^N C_i E_{B_i, x_i} \right) + \mathcal{D} \left(\sum_{j=1}^M \tilde{C}_j E_{A_j, y_j} \right). \end{aligned}$$

(iii) If $\mathcal{D} \left(\sum_{i=1}^N C_i E_{B_i, x_i} \right) = 0$, then $\left\| \sum_{i=1}^N C_i F(B_i) T(x_i) \right\|_Z = 0$, and hence $\sum_{i=1}^N C_i F(B_i) T(x_i) = 0$. This implies that for any $A \in \Sigma$, we have

$$\begin{aligned} \sum_{i=1}^N C_i E_{B_i, x_i}(A) &= \sum_{i=1}^N C_i E(B_i \cap A)(x_i) = \sum_{i=1}^N C_i S F(B_i \cap A) T(x_i) \\ &= S F(A) \sum_{i=1}^N C_i F(B_i) T(x_i) = 0. \end{aligned}$$

Thus \mathcal{D} is a norm on M_E .

Denote this norm by $\|\cdot\|_{\mathcal{D}}$, and let $\widetilde{M}_{E, \mathcal{D}}$ be the completion of $M_{E, \mathcal{D}}$ under this norm. We will show that \mathcal{D} is a dilation norm of E .

First, since

$$\begin{aligned} \left\| S_{\mathcal{D}} \left(\sum_{i=1}^N C_i E_{B_i, x_i} \right) \right\|_Y &= \left\| \sum_{i=1}^N C_i E(B_i) x_i \right\|_Y \\ &= \left\| \sum_{i=1}^N C_i S F(B_i) T x_i \right\|_Y = \left\| S \left(\sum_{i=1}^N C_i F(B_i) T x_i \right) \right\|_Y \\ &\leq \|S\| \cdot \left\| \sum_{i=1}^N C_i F(B_i) T x_i \right\|_Z = \|S\| \cdot \left\| \sum_{i=1}^N C_i E_{B_i, x_i} \right\|_{\mathcal{D}}, \end{aligned}$$

and

$$\begin{aligned} \|T_{\mathcal{D}}(x)\|_{\mathcal{D}} &= \|E_{\Omega, x}\|_{\mathcal{D}} = \|F(\Omega)T(x)\|_Z \\ &\leq \|F(\Omega)\| \cdot \|T\| \cdot \|x\|_X, \end{aligned}$$

we have that the mappings

$$S_{\mathcal{D}} : \widetilde{M}_{E, \mathcal{D}} \rightarrow Y, \quad S_{\mathcal{D}} \left(\sum_{i=1}^N C_i E_{B_i, x_i} \right) = \sum_{i=1}^N C_i E(B_i) x_i$$

and

$$T_{\mathcal{D}} : X \rightarrow \widetilde{M}_{E,\mathcal{D}}, \quad T_{\mathcal{D}}(x) = E_{\Omega,x}$$

are both well-defined, linear and bounded.

We prove that the mapping $F_{\mathcal{D}} : \Sigma \rightarrow B(\widetilde{M}_{E,\mathcal{D}})$ defined by

$$F_{\mathcal{D}}(B) \left(\sum_{i=1}^N C_i E_{B_i, x_i} \right) = \sum_{i=1}^N C_i E_{B \cap B_i, x_i},$$

is an operator-valued measure. By Lemma 2.25, we only need to show that $F_{\mathcal{D}}$ is strongly countably additive and uniformly bounded on $M_{E,\mathcal{D}}$.

If $\{A_j\}_{j=1}^{\infty}$ is a disjoint countable collection of members of Σ with union A , since F is an operator-valued measure, we have

$$F(A \cap B_i)T(x_i) = \sum_{j=1}^{\infty} F(A_j \cap B_i)T(x_i).$$

It follows that

$$F_{\mathcal{D}}(A)(E_{B_i, x_i}) = \sum_{j=1}^{\infty} F_{\mathcal{D}}(A_j)(E_{B_i, x_i}).$$

Hence

$$F_{\mathcal{D}}(A) \left(\sum_{i=1}^N C_i E_{B_i, x_i} \right) = \sum_{j=1}^{\infty} F_{\mathcal{D}}(A_j) \left(\sum_{i=1}^N C_i E_{B_i, x_i} \right).$$

Thus $F_{\mathcal{D}}$ is strongly countably additive on $M_{E,\mathcal{D}}$.

For any $A \in \Sigma$, we have

$$\begin{aligned} & \left\| F_{\mathcal{D}}(A) \left(\sum_{i=1}^N C_i E_{B_i, x_i} \right) \right\|_{\mathcal{D}} = \left\| \sum_{i=1}^N C_i F_{\mathcal{D}}(A)(E_{B_i, x_i}) \right\|_{\mathcal{D}} \\ &= \left\| \sum_{i=1}^N C_i E_{A \cap B_i, x_i} \right\|_{\mathcal{D}} = \left\| \sum_{i=1}^N C_i F(A \cap B_i)T(x_i) \right\|_Z \\ &= \left\| \sum_{i=1}^N C_i F(A)F(B_i)T(x_i) \right\|_Z \leq \|F(A)\| \left\| \sum_{i=1}^N C_i F(B_i)T(x_i) \right\|_Z \\ &\leq \sup_{A \in \Sigma} \|F(A)\| \left\| \sum_{i=1}^N C_i E_{B_i, x_i} \right\|_{\mathcal{D}}. \end{aligned}$$

By Remark 2.4, we know that

$$\sup_{A \in \Sigma} \|F(A)\| < +\infty,$$

and thus $F_{\mathcal{D}}$ is uniformly bounded on $M_{E,\mathcal{D}}$.

Define a mapping $U : \widetilde{M}_{E,\mathcal{D}} \rightarrow Z$ by

$$U \left(\sum_{i=1}^N C_i E_{B_i, x_i} \right) = \sum_{i=1}^N C_i F(B_i)T(x_i).$$

It is easy to see that U is a well-defined and linear isometric embedding mapping.

Finally, we have

$$\begin{aligned} S_{\mathcal{D}} \left(\sum_{i=1}^N C_i E_{B_i, x_i} \right) &= \sum_{i=1}^N C_i E(B_i) x_i = \sum_{i=1}^N C_i S F(B_i) T(x_i) \\ &= S \sum_{i=1}^N C_i F(B_i) T(x_i) = S U \left(\sum_{i=1}^N C_i E_{B_i, x_i} \right), \end{aligned}$$

$$U T_{\mathcal{D}}(x) = U(E_{\Omega, x}) = F(\Omega) T(x)$$

and

$$\begin{aligned} U F_{\mathcal{D}}(B) \left(\sum_{i=1}^N C_i E_{B_i, x_i} \right) &= U \left(\sum_{i=1}^N C_i E_{B \cap B_i, x_i} \right) = \sum_{i=1}^N C_i F(B \cap B_i) T(x_i) \\ &= \sum_{i=1}^N C_i F(B) F(B_i) T(x_i) = F(B) \left(\sum_{i=1}^N C_i F(B_i) T(x_i) \right) \\ &= F(B) U \left(\sum_{i=1}^N C_i E_{B_i, x_i} \right). \end{aligned}$$

Thus we get

$$S_{\mathcal{D}} = S U, \quad F(\Omega) T = U T_{\mathcal{D}}, \quad U F_{\mathcal{D}}(B) = F(B) U, \quad \forall B \in \Sigma,$$

as claimed. \square

In general, we can find different dilation norms on M_E , and hence E may have different dilation projection-valued measure systems. The following result shows that if E is a projection $B(X)$ -valued probability measure, then these different dilation projection-valued measure systems are all similar, that is, E is isomorphic to $F_{\mathcal{D}}$ for every dilation norm \mathcal{D} on M_E .

PROPOSITION 2.27. Let $(\Omega, \Sigma, E, B(X))$ be a projection-valued probability measure system and \mathcal{D} be a dilation norm of E . Then $S_{\mathcal{D}}$ and $T_{\mathcal{D}}$ both are invertible operators with $S_{\mathcal{D}} T_{\mathcal{D}} = I_X$ and $T_{\mathcal{D}} S_{\mathcal{D}} = I_{\widetilde{M}_{E, \mathcal{D}}}$. Thus, E is isomorphic to $F_{\mathcal{D}}$ for every dilation norm \mathcal{D} of E .

PROOF. By Theorem 2.23, we know that $(\Omega, \Sigma, F_{\mathcal{D}}, B(\widetilde{M}_{E, \mathcal{D}}), S_{\mathcal{D}}, T_{\mathcal{D}})$ is a projection-valued probability measure space system. Hence $F_{\mathcal{D}}(\Omega) = I_{\widetilde{M}_{E, \mathcal{D}}}$ and

$$S_{\mathcal{D}} T_{\mathcal{D}} = S_{\mathcal{D}} F_{\mathcal{D}}(\Omega) T_{\mathcal{D}} = E(\Omega) = I_X.$$

By Remark 2.14, we know that $S_{\mathcal{D}}$ is a surjection and $T_{\mathcal{D}}$ is an isomorphic embedding.

Let $N > 0$, $\{C_i\}_{i=1}^N \subset \mathbb{C}$, $\{x_i\}_{i=1}^N \subset X$ and $\{B_i\}_{i=1}^N \subset \Sigma$. If

$$S_{\mathcal{D}} \left(\sum_{i=1}^N C_i E_{B_i, x_i} \right) = \sum_{i=1}^N C_i E(B_i) x_i = 0,$$

then for any $A \in \Sigma$,

$$\sum_{i=1}^N C_i E_{B_i, x_i}(A) = \sum_{i=1}^N C_i E(B_i \cap A)(x_i) = E(A) \left(\sum_{i=1}^N C_i E(B_i) x_i \right) = 0.$$

So we get $\sum_{i=1}^N C_i E_{B_i, x_i} = 0$, and hence $S_{\mathcal{D}}$ is invertible.

Observe that for any $A \in \Sigma$,

$$E_{\Omega, \sum_{i=1}^N C_i E(B_i) x_i}(A) = E(A) \left(\sum_{i=1}^N C_i E(B_i) x_i \right) = \sum_{i=1}^N C_i E_{B_i, x_i}(A).$$

Thus we have

$$\begin{aligned} T_{\mathcal{D}} S_{\mathcal{D}} \left(\sum_{i=1}^N C_i E_{B_i, x_i} \right) &= T_{\mathcal{D}} \left(\sum_{i=1}^N C_i E(B_i) x_i \right) \\ &= E_{\Omega, \sum_{i=1}^N C_i E(B_i) x_i} = \sum_{i=1}^N C_i E_{B_i, x_i}. \end{aligned}$$

So we get $T_{\mathcal{D}} S_{\mathcal{D}} = I_{\widetilde{M}_{\varphi, \mathcal{D}}}$ and $T_{\mathcal{D}}$ is invertible. Therefore E is isomorphic to $F_{\mathcal{D}}$ for every dilation norm \mathcal{D} of E . \square

2.4. The Minimal Dilation Norm $\|\cdot\|_{\alpha}$

While the connection between a corresponding dilation projection-valued measure system and the elementary dilation space has been established in the previous section, we still need to address the existence issue for a corresponding dilation projection-valued measure system. So in this and the next two sections we will focus on constructing two special dilation norms for the (algebraic) elementary dilation space M_E which we call the minimal and maximal dilation norms.

DEFINITION 2.28. Define $\|\cdot\|_{\alpha} : M_E \rightarrow \mathbb{R}^+ \cup \{0\}$ by

$$\left\| \sum_{i=1}^N C_i E_{B_i, x_i} \right\|_{\alpha} = \sup_{B \in \Sigma} \left\| \sum_{i=1}^N C_i E(B \cap B_i) x_i \right\|_Y.$$

PROPOSITION 2.29. $\|\cdot\|_{\alpha}$ is a norm on M_E .

PROOF. Let $N > 0$, $\{C_i\}_{i=1}^N \subset \mathbb{C}$, $\{x_i\}_{i=1}^N \subset X$ and $\{B_i\}_{i=1}^N \subset \Sigma$. If $\sum_{i=1}^N C_i E_{B_i, x_i} = 0$, then for any $B \in \Sigma$, $\sum_{i=1}^N C_i E(B \cap B_i) x_i = 0$, hence

$$\left\| \sum_{i=1}^N C_i E_{B_i, x_i} \right\|_{\alpha} = \sup_{B \in \Sigma} \left\| \sum_{i=1}^N C_i E(B \cap B_i) x_i \right\|_Y = 0.$$

Thus $\|\cdot\|_{\alpha}$ is well-defined.

If $\left\| \sum_{i=1}^N C_i E_{B_i, x_i} \right\|_{\alpha} = 0$, then for any $B \in \Sigma$, $\sum_{i=1}^N C_i E(B \cap B_i) x_i = 0$, thus

$$\sum_{i=1}^N C_i E_{B_i, x_i} = 0.$$

Now we prove that $\|\cdot\|_\alpha$ satisfies the triangle inequality. Let $M > 0$, $\{\tilde{C}_j\}_{j=1}^M \subset \mathbb{C}$, $\{y_j\}_{j=1}^M \subset X$ and $\{A_j\}_{j=1}^M \subset \Sigma$. Then

$$\begin{aligned}
& \left\| \sum_{i=1}^N C_i E_{B_i, x_i} + \sum_{j=1}^M \tilde{C}_j E_{A_j, y_j} \right\|_\alpha \\
&= \sup_{B \in \Sigma} \left\| \sum_{i=1}^N C_i E(B_i \cap B) x_i + \sum_{j=1}^M \tilde{C}_j E(A_j \cap B) y_j \right\|_Y \\
&\leq \sup_{B \in \Sigma} \left\| \sum_{i=1}^N C_i E(B_i \cap B) x_i \right\|_Y + \sup_{B \in \Sigma} \left\| \sum_{j=1}^M \tilde{C}_j E(A_j \cap B) y_j \right\|_Y \\
&= \left\| \sum_{i=1}^N C_i E_{B_i, x_i} \right\|_\alpha + \left\| \sum_{j=1}^M \tilde{C}_j E_{A_j, y_j} \right\|_\alpha.
\end{aligned}$$

So $\|\cdot\|_\alpha$ is a norm on M_E . \square

We denote its normed space by $M_{E, \alpha}$ and its completion by $\widetilde{M}_{E, \alpha}$.

THEOREM 2.30. $\|\cdot\|_\alpha$ is a dilation norm of E .

PROOF. Let $N > 0$, $\{C_i\}_{i=1}^N \subset \mathbb{C}$, $\{x_i\}_{i=1}^N \subset X$ and $\{B_i\}_{i=1}^N \subset \Sigma$. Since the linear map $S_\alpha : \widetilde{M}_{E, \alpha} \rightarrow Y$ is defined by

$$S_\alpha \left(\sum_{i=1}^N C_i E_{B_i, x_i} \right) = \sum_{i=1}^N C_i E(B_i) x_i,$$

we have that

$$\begin{aligned}
& \left\| S_\alpha \left(\sum_{i=1}^N C_i E_{B_i, x_i} \right) \right\|_Y = \left\| \sum_{i=1}^N C_i E(B_i) x_i \right\|_Y \\
&\leq \sup_{B \in \Sigma} \left\| \sum_{i=1}^N C_i E(B \cap B_i) x_i \right\|_Y = \left\| \sum_{i=1}^N C_i E_{B_i, x_i} \right\|_{\mathcal{M}},
\end{aligned}$$

which implies that S_α is bounded and $\|S_\alpha\| \leq 1$.

Recall that the map $T_\alpha : X \rightarrow \widetilde{M}_{E, \alpha}$ is defined by $T_\alpha(x) = E_{\Omega, x}$. Obviously, we have

$$\|T_\alpha x\|_\alpha = \|E_{\Omega, x}\|_\alpha = \sup_{B \in \Sigma} \|E(B)x\|_Y \leq \|E\| \cdot \|x\|,$$

and hence $\|T_\alpha\| \leq \|E\|$.

Now we prove the mapping $F_\alpha : \Sigma \rightarrow B(\widetilde{M}_{E, \alpha})$ defined by

$$F_\alpha(B) \left(\sum_{i=1}^N C_i E_{B_i, x_i} \right) = \sum_{i=1}^N C_i E_{B \cap B_i, x_i}$$

is an operator-valued measure. By Lemma 2.25, we only need to prove F_α is strongly countably additive and uniform bounded on $M_{E, \alpha}$.

Since

$$\begin{aligned}
& \left\| F_\alpha(B) \left(\sum_{i=1}^N C_i E_{B_i, x_i} \right) \right\|_\alpha = \left\| \sum_{i=1}^N C_i E_{B \cap B_i, x_i} \right\|_\alpha \\
&= \sup_{B' \in \Sigma} \left\| \sum_{i=1}^N C_i E(B_i \cap B \cap B')(x_i) \right\|_Y = \sup_{B' \in \Sigma} \left\| \sum_{i=1}^N C_i E(B_i \cap B') x_i \right\|_Y \\
&= \left\| \sum_{i=1}^N C_i E_{B_i, x_i} \right\|_Y,
\end{aligned}$$

we get that $\|F_\alpha(B)\| = 1$ for any $B \in \Sigma$.

For the strong countable additivity, let $\{A_j\}_{j=1}^\infty$ be a countable disjoint collection of members of Σ with union A . Then we have

$$\begin{aligned}
& \left\| \sum_{j=1}^M F_\alpha(A_j) \left(\sum_{i=1}^N C_i E_{B_i, x_i} \right) - F_\alpha(A) \left(\sum_{i=1}^N C_i E_{B_i, x_i} \right) \right\|_\alpha \\
&= \left\| \sum_{j=1}^M \left(\sum_{i=1}^N C_i E_{A_j \cap B_i, x_i} \right) - \left(\sum_{i=1}^N C_i E_{A \cap B_i, x_i} \right) \right\|_\alpha \\
&= \left\| \sum_{i=1}^N C_i \left(\sum_{j=1}^M E_{A_j \cap B_i, x_i} - E_{A \cap B_i, x_i} \right) \right\|_\alpha \\
&= \sup_{B' \in \Sigma} \left\| \sum_{i=1}^N C_i \left(\sum_{j=1}^M E(A_j \cap B_i \cap B') x_i - E(A \cap B_i \cap B') x_i \right) \right\|_Y \\
&= \sup_{B' \in \Sigma} \left\| \sum_{i=1}^N C_i E \left(\bigcup_{j=M+1}^\infty (A_j \cap B_i \cap B') \right) x_i \right\|_Y \\
&\leq \sum_{i=1}^N |C_i| \sup_{B' \in \Sigma} \left\| E \left(\bigcup_{j=M+1}^\infty (A_j \cap B_i \cap B') \right) x_i \right\|_Y \\
&= \sum_{i=1}^N |C_i| \sup_{B' \in \Sigma} \left\| E \left(\bigcup_{j=M+1}^\infty (A_j \cap B') \right) x_i \right\|_Y \\
&= \sum_{i=1}^N |C_i| \sup_{B' \in \Sigma} \left\| \sum_{j=M+1}^\infty E(A_j \cap B') x_i \right\|_Y.
\end{aligned}$$

If $\sup_{B' \in \Sigma} \left\| \sum_{j=M+1}^\infty E(A_j \cap B') x_i \right\|_Y$ does not tend to 0 as $M \rightarrow \infty$, then we can find $\delta > 0$, a sequence of $n_1 \leq m_1 < n_2 \leq m_2 < n_3 \leq m_3 < \dots$, and $\{B'_l\}_{l=1}^\infty \subset \Sigma$ such that

$$\left\| \sum_{j=n_l}^{m_l} E(A_j \cap B'_l) x_i \right\|_Y \geq \delta, \quad \forall l \in \mathbb{N}.$$

Since for $l \in \mathbb{N}$ and $n_l \leq j \leq m_l$, $A_j \cap B'_l$ are disjoint from each other, we have

$$E\left(\bigcup_{l=1}^{\infty} \bigcup_{j=n_l}^{m_l} A_j \cap B'_l\right) x_i = \sum_{l=1}^{\infty} \sum_{j=n_l}^{m_l} E(B_j \cap B'_l) x_i,$$

which implies $\left\|\sum_{j=n_i}^{m_i} E(B_j \cap B'_i) x_i\right\| \rightarrow 0$, which is a contradiction. Hence

$$F_\alpha(A) \left(\sum_{i=1}^N C_i E_{B_i, x_i}\right) = \sum_{j=1}^{\infty} F_\alpha(A_j) \left(\sum_{i=1}^N C_i E_{B_i, x_i}\right),$$

as expected. \square

Combining Theorem 2.30 and Theorem 2.23 we have the following:

THEOREM 2.31. *For any operator-valued measure system $(\Omega, \Sigma, E, B(X, Y))$, there exist a Banach space Z , two bounded linear operators $S : Z \rightarrow Y$ and $T : X \rightarrow Z$, and a projection-valued probability measure system $(\Omega, \Sigma, F, B(Z))$ such that $E(B) = SF(B)T$ for any $B \in \Sigma$. In other words, every operator-valued measure can be dilated to a projection-valued measure.*

LEMMA 2.32. *Let $(\Omega, \Sigma, E, B(X, Y))$ be an operator-valued measure system. If \mathcal{D} is a dilation norm of E , then there exists a constant $C_{\mathcal{D}}$ such that for any $\sum_{i=1}^N C_i E_{B_i, x_i} \in M_{E, \mathcal{D}}$,*

$$\sup_{B \in \Sigma} \left\| \sum_{i=1}^N C_i E(B \cap B_i) x_i \right\|_Y \leq C_{\mathcal{D}} \left\| \sum_{i=1}^N C_i E_{B_i, x_i} \right\|_{\mathcal{D}},$$

where $N > 0$, $\{C_i\}_{i=1}^N \subset \mathbb{C}$, $\{x_i\}_{i=1}^N \subset X$ and $\{B_i\}_{i=1}^N \subset \Sigma$. Consequently we have that

$$\|f\|_\alpha \leq C_{\mathcal{D}} \|f\|_{\mathcal{D}}, \quad \forall f \in M_E.$$

PROOF. Let $C_{\mathcal{D}} = \sup_{B \in \Sigma} \|S_{\mathcal{D}} F_{\mathcal{D}}(B)\|$. Then obviously $C_{\mathcal{D}} < +\infty$. We also have

$$\begin{aligned} \sup_{B \in \Sigma} \left\| \sum_{i=1}^N C_i E(B \cap B_i) x_i \right\|_Y &= \sup_{B \in \Sigma} \left\| \sum_{i=1}^N C_i S_{\mathcal{D}} F_{\mathcal{D}}(B \cap B_i) T_{\mathcal{D}}(x_i) \right\|_Y \\ &= \sup_{B \in \Sigma} \left\| \sum_{i=1}^N C_i S_{\mathcal{D}} F_{\mathcal{D}}(B) F_{\mathcal{D}}(B_i) T_{\mathcal{D}}(x_i) \right\|_Y \\ &\leq \sup_{B \in \Sigma} \|S_{\mathcal{D}} F_{\mathcal{D}}(B)\| \left\| \sum_{i=1}^N C_i F_{\mathcal{D}}(B_i) T_{\mathcal{D}}(x_i) \right\|_{\mathcal{D}} \\ &= C_{\mathcal{D}} \left\| \sum_{i=1}^N C_i E_{B_i, x_i} \right\|_{\mathcal{D}}. \end{aligned}$$

\square

This lemma justifies the following definition for the dilation norm α .

DEFINITION 2.33. Let $(\Omega, \Sigma, E, B(X, Y))$ be an operator-valued measure system. We call α the *minimal dilation norm* of E and $\widetilde{M}_{E, \alpha}$ the α -dilation space.

The next result shows that if we start with a projection-valued probability measure E , then the dilated projection-valued probability measure on the elementary dilation space $\widetilde{M}_{E,\alpha}$ is isometric to E .

PROPOSITION 2.34. Let $(\Omega, \Sigma, E, B(X))$ be a normalized projection-valued probability measure system and α be the minimal dilation norm of E . Then S_α and T_α both are isometries and E is isometric to F_α .

PROOF. Since α is a dilation norm of E , by Proposition 2.27, S_α and T_α both are invertible operators with $S_\alpha T_\alpha = I_X$ and $T_\alpha S_\alpha = I_{\widetilde{M}_{E,\alpha}}$. For any $x \in X$, we have

$$\|T_\alpha x\|_\alpha = \|E_{\Omega,x}\|_\alpha = \sup_{B \in \Sigma} \|E(B)x\| \leq \sup_{B \in \Sigma} \|E(B)\| \|x\| = \|x\|,$$

and thus $\|T_\alpha\| \leq 1$.

On the other hand, for any $N > 0$, $\{C_i\}_{i=1}^N \subset \mathbb{C}$, $\{x_i\}_{i=1}^N \subset X$ and $\{B_i\}_{i=1}^N \subset \Sigma$, we have that

$$\begin{aligned} & \left\| S_\alpha \left(\sum_{i=1}^N C_i E_{B_i, x_i} \right) \right\|_X = \left\| \sum_{i=1}^N C_i E(B_i) x_i \right\|_X \\ & \leq \sup_{B' \in \Sigma} \left\| \sum_{i=1}^N C_i E(B_i \cap B') x_i \right\|_X = \left\| \sum_{i=1}^N C_i E_{B_i, x_i} \right\|_\alpha, \end{aligned}$$

and so $\|S_\alpha\| \leq 1$.

Since $S_\alpha T_\alpha = I_X$ and $T_\alpha S_\alpha = I_{\widetilde{M}_{E,\alpha}}$, we have that for each $x \in X$ we have

$$\|x\| = \|S_\alpha T_\alpha x\| \leq \|T_\alpha x\| \leq \|x\|.$$

Hence T_α is a surjective isometry. Similarly S_α is also a surjective isometry, and therefore E is isometric to F_α . \square

PROPOSITION 2.35. Let $(\Omega, \Sigma, E, B(X, Y))$ be an operator-valued measure system and α be the minimal dilation norm of E . If $P_j : Y \rightarrow Y$ are norm one projections such that $\sum_{j=1}^M P_j = I_Y$, then

$$E_j(B) : \Sigma \rightarrow B(X, Y), \quad E_j(B)x = P_j E(B)x, \quad x \in X$$

is an operator-valued measure for $1 \leq j \leq M$. Moreover for any $\sum_{i=1}^N C_i E_{B_i, x_i} \in M_E$, we have

$$\max_{1 \leq j \leq M} \left\| \sum_{i=1}^N C_i (E_j)_{B_i, x_i} \right\|_{\alpha_j} \leq \left\| \sum_{i=1}^N C_i E_{B_i, x_i} \right\|_\alpha \leq \sum_{1 \leq j \leq M} \left\| \sum_{i=1}^N C_i (E_j)_{B_i, x_i} \right\|_{\alpha_j},$$

where α_j is the minimal dilation norm of E_j .

PROOF. It is obvious that for each $1 \leq j \leq M$,

$$E_j(B) : \Sigma \rightarrow B(X, Y), \quad E_j(B)x = P_j E(B)x, \quad x \in X$$

is an operator-valued measure.

The “moreover” part follows from:

$$\begin{aligned}
\left\| \sum_{i=1}^N C_i(E_j)_{B_i, x_i} \right\|_{\alpha_j} &= \sup_{B' \in \Sigma} \left\| \sum_{i=1}^N C_i P_j E(B_i \cap B') x_i \right\| \\
&= \sup_{B' \in \Sigma} \left\| P_j \left(\sum_{i=1}^N C_i E(B_i \cap B') x_i \right) \right\| \\
&\leq \|P_j\| \sup_{B' \in \Sigma} \left\| \sum_{i=1}^N C_i E(B_i \cap B') x_i \right\| \\
&= \sup_{B' \in \Sigma} \left\| \sum_{i=1}^N C_i E(B_i \cap B') x_i \right\| \\
&= \sup_{B' \in \Sigma} \left\| \sum_{j=1}^M P_j \left(\sum_{i=1}^N C_i E(B_i \cap B') x_i \right) \right\| \\
&\leq \sum_{j=1}^M \sup_{B' \in \Sigma} \left\| \sum_{i=1}^N C_i P_j (E(B_i \cap B') x_i) \right\| \\
&= \sum_{1 \leq j \leq M} \left\| \sum_{i=1}^N C_i(E_j)_{B_i, x_i} \right\|_{\alpha_j}.
\end{aligned}$$

□

LEMMA 2.36. *If $\{i\}$ is an atom in Σ , then the rank of $F_\alpha(\{i\})$ is equal to the rank of $E(\{i\})$.*

PROOF. Suppose that the rank of $E(\{i\})$ is k . Then there exist $x_1, \dots, x_k \in X$ such that $\text{range } E(\{i\}) = \text{span}\{E(\{i\})x_1, \dots, E(\{i\})x_k\}$. We show that $\text{range } F(\{i\}) = \text{span}\{E_{x_1, \{i\}}, \dots, E_{x_k, \{i\}}\}$. In fact, for any $x \in X, B \in \Sigma$, we have that $F(\{i\})E_{B, x} = E_{\{i\}, x}$ or 0. Now for any $A \in \Sigma$, we have $E_{\{i\}, x}(A) = E(\{i\})x$ if $i \in A$ and 0 if $i \notin A$. Write $E(\{i\})x = \sum_{j=1}^k c_j E(\{i\})x_j$. Then we get $E_{\{i\}, x}(A) = \sum_{j=1}^k c_j E(\{i\})x_j = \sum_{j=1}^k c_j E_{\{i\}, x_j}(A)$ if $i \in A$ and 0 otherwise. Hence

$$E_{\{i\}, x} = \sum_{j=1}^k c_j E_{\{i\}, x_j},$$

and therefore $\text{range } F(\{i\}) = \text{span}\{E_{\{i\}, x_1}, \dots, E_{\{i\}, x_k}\}$ as claimed. □

Problem A. Is it always true that with an appropriate notion of rank function for an operator valued measure, that $r(F(B)) = r(E(B))$ for every $B \in \Sigma$? The previous lemma tells us that it is true when B is an atom. A possible “rank” definition might be: $r(B) = \sup\{\text{rank } E(A) : A \subset B, A \in \Sigma\}$.

EXAMPLE 2.37. Assume that $(\mathbb{N}, 2^\mathbb{N}, E, B(X))$ is an induced operator-valued probability measure system by a framing $\{x_i, y_i\}_{i \in \mathbb{N}}$ of X , where $E(B) = \sum_{i \in B} x_i \otimes y_i$ for every $B \in 2^\mathbb{N}$.

We characterize its minimal α -dilation space $\widetilde{M}_{E,\alpha}$. Let $(\mathbb{N}, 2^{\mathbb{N}}, F_\alpha, B(\widetilde{M}_{E,\alpha}))$ be the corresponding spectral operator-valued probability measure. Then we have

$$f = \sum_{i \in \mathbb{N}} F_\alpha(\{i\})f, \quad \forall f \in \widetilde{M}_{E,\alpha}.$$

By Lemma 2.36, all $F_\alpha(\{i\})$'s are rank-one projections. Choose \tilde{x}_i such that $y_i(\tilde{x}_i) = 1$. Then

$$F_\alpha(\{i\})(E_{\{i\}, \tilde{x}_i}) = E_{\{i\}, \tilde{x}_i}, \quad \|E_{\{i\}, \tilde{x}_i}\|_\alpha = \|x_i\|, \quad \forall i \in \mathbb{N}.$$

And because $F_\alpha(\{i\})$'s are projections with $F_\alpha(\{i\})F_\alpha(\{j\}) = \delta_{ij}F_\alpha(\{i\})$, we know that $\{E_{\{i\}, \tilde{x}_i}\}$ is a basis of $\widetilde{M}_{E,\alpha}$, which is just our minimal framing model basis. Actually, for every $(a_i) \in c_{00}$, we have

$$\begin{aligned} \left\| \sum_{i \in \mathbb{N}} a_i E_{\{i\}, \tilde{x}_i} \right\|_\alpha &= \sup_{B \subset \mathbb{N}} \left\| \sum_{i \in \mathbb{N}} a_i E(\{i\} \cap B) \tilde{x}_i \right\| \\ &= \sup_{B \subset \mathbb{N}} \left\| \sum_{i \in B} a_i E(\{i\}) \tilde{x}_i \right\| \\ &= \sup_{B \subset \mathbb{N}} \left\| \sum_{i \in B} a_i (x_i \otimes y_i) \tilde{x}_i \right\| \\ &= \sup_{B \subset \mathbb{N}} \left\| \sum_{i \in B} a_i y_i(\tilde{x}_i) x_i \right\| \\ &= \sup_{B \subset \mathbb{N}} \left\| \sum_{i \in B} a_i x_i \right\|, \end{aligned}$$

which is equivalent to the minimal framing model basis because of the following inequality.

$$\frac{1}{2} \sup_{\epsilon_i = \pm 1} \left\| \sum_{i \in \mathbb{N}} \epsilon_i a_i x_i \right\| \leq \sup_{B \subset \mathbb{N}} \left\| \sum_{i \in B} a_i x_i \right\| \leq \sup_{\epsilon_i = \pm 1} \left\| \sum_{i \in \mathbb{N}} \epsilon_i a_i x_i \right\|.$$

This is because

$$\begin{aligned}
\frac{1}{2} \sup_{\epsilon_i = \pm 1} \left\| \sum_{i \in \mathbb{N}} \epsilon_i a_i x_i \right\| &= \frac{1}{2} \sup_{\epsilon_i = \pm 1} \left\| \sum_{\substack{i \in \mathbb{N} \\ \epsilon_i = 1}} a_i x_i - \sum_{\substack{i \in \mathbb{N} \\ \epsilon_i = -1}} a_i x_i \right\| \\
&\leq \frac{1}{2} \sup_{\epsilon_i = \pm 1} \left(\sup_{B \subset \mathbb{N}} \left\| \sum_{i \in B} a_i x_i \right\| + \sup_{B \subset \mathbb{N}} \left\| \sum_{i \in B} a_i x_i \right\| \right) \\
&= \sup_{B \subset \mathbb{N}} \left\| \sum_{i \in B} a_i x_i \right\| \\
&= \frac{1}{2} \sup_{B \subset \mathbb{N}} \left\| \sum_{\substack{i \in \mathbb{N} \\ \epsilon_i = 1}} \epsilon_i a_i x_i + \sum_{\substack{\epsilon_i = 1, i \in B \\ \epsilon_i = -1, i \notin B}} \epsilon_i a_i x_i \right\| \\
&\leq \sup_{B \subset \mathbb{N}} \sup_{\epsilon_i = \pm 1} \left\| \sum_{i \in \mathbb{N}} \epsilon_i a_i x_i \right\| \\
&= \sup_{\epsilon_i = \pm 1} \left\| \sum_{i \in \mathbb{N}} \epsilon_i a_i x_i \right\|.
\end{aligned}$$

The above example shows that in the case that the operator-valued probability measure system is induced by a framing $\{x_i, y_i\}_{i \in \mathbb{N}}$ of X , then the minimal α -dilation space is exactly the one constructed in [CHL] and hence it is always separable. However, it is not clear whether this is true in general. So we ask:

Problem B. Assume that $(\Omega, \Sigma, E, B(X))$ is an operator-valued measure system such that X is a separable Banach space. Is the dilation space α equipped with the minimal dilation norm always separable? If not, does there exist a dilation space which is separable?

We end this section with a result concerning this problem. We say that an operator valued measure E on (Ω, Σ) into $B(X)$ is *totally strongly countably additive* if whenever $\{B_n\}$ is a disjoint sequence in Σ with union B , and $x \in X$, then given any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\sup_{A \in \Sigma} \left\| \sum_{j=n}^{\infty} E(B_j \cap A)x \right\| < \epsilon$$

holds for all $n \geq N$. We say that (Ω, Σ, E) has a *countable Borel basis* if there exists a countable set \mathcal{B} of subsets from Σ such that for each $B \in \Sigma$, $B = \sum_{n=1}^{\infty} B_n$ for some subsets B_n from \mathcal{B} up to a E -null set (A E -null set is a set $C \in \Sigma$ such that $E(A) = 0$ if $A \subseteq C$ and $A \in \Sigma$).

PROPOSITION 2.38. Assume that an operator valued measure E on (Ω, Σ) into $B(X)$ is totally strongly countably additive, and that (Ω, Σ, E) has a countable Borel basis. If X is separable, then so is the dilation space α with the minimal dilation norm.

PROOF. Fix $x \in X$, and let $\{B_i\}$ be a countable Borel basis for (Ω, Σ, E) . Then it is easy to prove that $\overline{\text{span}}\{E_{B_i, x} : i \in \mathbb{N}\}$ contains $E_{B, x}$ for every $B \in \Sigma$.

Now fix $B \in \Sigma$, and let $\{x_i\}$ be a dense subset of X . Then $\overline{\text{span}}\{E_{B,x_i} : i \in \mathbb{N}\}$ contains $E_{B,x}$ for every $x \in X$. Therefore we get that $\text{span}\{E_{B_i,x_j} : i, j \in \mathbb{N}\}$ is dense in α and hence α is separable. \square

2.5. The Dual Space of the Minimal Dilation Space

In this section we give a concrete description for the dual space of the α -dilation space. Let $(\Omega, \Sigma, E, B(X, Y))$ be an operator-valued measure system, Y^* be the dual space of Y and $\widetilde{M}_{E,\alpha}^*$ be the dual space of $\widetilde{M}_{E,\alpha}$. For any $y^* \in Y^*$ and $B \in \Sigma$, define a mapping $\phi_{B,y^*} : \widetilde{M}_{E,\alpha} \rightarrow \mathbb{C}$ by

$$\phi_{B,y^*} \left(\sum_{i=1}^N C_i E_{B_i,x_i} \right) = y^* \left(\sum_{i=1}^N C_i E(B \cap B_i)(x_i) \right).$$

If $\sum_{i=1}^N C_i E_{B_i,x_i} = 0$, then

$$\phi_{B,y^*} \left(\sum_{i=1}^N C_i E_{B_i,x_i} \right) = y^* \left(\sum_{i=1}^N C_i E(B \cap B_i)(x_i) \right) = y^* \left(\sum_{i=1}^N C_i E_{B_i,x_i}(B) \right) = 0.$$

Thus the mapping $\phi_{B,y^*} : \widetilde{M}_{E,\alpha} \rightarrow \mathbb{C}$ is well-defined.

PROPOSITION 2.39. Let $\mathcal{M}_E = \text{span}\{\phi_{B,y^*} : y^* \in Y^*, B \in \Sigma\}$. Then $\mathcal{M}_E \subset \widetilde{M}_{E,\alpha}^*$.

PROOF. It is sufficient to prove that ϕ_{B,y^*} is bounded on $M_{E,\alpha}$. In fact, this follows immediately from the following:

$$\begin{aligned} & \left| \phi_{B,y^*} \left(\sum_{i=1}^N C_i E_{B_i,x_i} \right) \right| \\ &= \left| y^* \left(\sum_{i=1}^N C_i E(B \cap B_i)(x_i) \right) \right| \\ &\leq \|y^*\| \left\| \sum_{i=1}^N C_i E(B_i \cap B)x_i \right\| \\ &\leq \|y^*\| \left\| \sum_{i=1}^N C_i E_{B_i,x_i} \right\|_{\alpha}. \end{aligned}$$

\square

We will denote the norm on $\widetilde{M}_{E,\alpha}^*$ by α^* . Then the space \mathcal{M}_E endowed with the norm α^* is a normed space, which will be denoted by \mathcal{M}_{E,α^*} .

PROPOSITION 2.40. For every $\sum_{i=1}^N C_i \phi_{B_i,y_i^*} \in \mathcal{M}_{E,\alpha^*}$, we have

$$\left\| \sum_{i=1}^N C_i \phi_{B_i,y_i^*} \right\|_{\alpha^*} \leq \sum_{i=1}^N |C_i| \|y_i^*\|.$$

PROOF. First, for every $\phi_{y^*,B} \in \mathcal{M}_{E,\alpha^*}$, by the proof of Proposition 2.39, we have

$$\|\phi_{B,y^*}\| \leq \|y^*\|.$$

Thus, for every $\sum_{i=1}^N C_i \phi_{B_i, y_i^*} \in \mathcal{M}_{E, \alpha^*}$, we get

$$\left\| \sum_{i=1}^N C_i \phi_{B_i, y_i^*} \right\|_{\alpha^*} \leq \sum_{i=1}^N |C_i| \|\phi_{B_i, y_i^*}\| \leq \sum_{i=1}^N |C_i| \|y_i^*\|.$$

□

Since α^* is a norm on \mathcal{M}_E , we denote the completion of \mathcal{M}_E under norm α^* by $\widetilde{\mathcal{M}}_{E, \alpha^*}$. Then $\widetilde{\mathcal{M}}_{E, \alpha^*}$ is a Banach space, and so $\mathcal{M}_{E, \alpha^*} \subset \widetilde{\mathcal{M}}_{E, \alpha^*}^*$.

PROPOSITION 2.41. $\widetilde{M}_{E, \alpha}^* = \overline{\widetilde{\mathcal{M}}_{E, \alpha^*}}^{w^*}$.

PROOF. By the Hahn-Banach Separation Theorem with respect to the w^* -topology, it is enough to show that $\widetilde{\mathcal{M}}_{E, \alpha^*}$ separates $\widetilde{M}_{E, \alpha}$. If not, then there is an $f \in \widetilde{M}_{E, \alpha}$ with $\|f\|_\alpha = 1$ such that $g^*(f) = 0$ for every $g^* \in \widetilde{M}_{E, \alpha}^*$. For any $0 < \epsilon < 1/2$, we can find a $g \in M_{E, \alpha}$ with $\|g\|_\alpha = 1$ and $\|f - g\|_\alpha \leq \epsilon$. Let $g = \sum_{i=1}^N C_i E_{B_i, x_i}$ be a representation of g . By the definition of α -dilation norm, there is $E \in \Sigma$ satisfying

$$\left\| \sum_{i=1}^N C_i E(B_i \cap B)(x_i) \right\| \geq 1 - \epsilon.$$

Take an $y^* \in Y^*$ such that

$$y^* \left(\sum_{i=1}^N C_i E(B_i \cap B)x_i \right) = \left\| \sum_{i=1}^N C_i E(B_i \cap B)x_i \right\| \geq 1 - \epsilon.$$

Then for $\phi_{B, y^*} \in \mathcal{M}_{E, \alpha^*}$,

$$\phi_{B, y^*}(g) = \phi_{B, y^*} \left(\sum_{i=1}^N C_i E_{B_i, x_i} \right) = y^* \left(\sum_{i=1}^N C_i E(B_i \cap B)x_i \right) \geq 1 - \epsilon$$

and $\|\phi_{B, y^*}\|_{\alpha^*} \leq \|y^*\| = 1$. Thus, we have

$$\begin{aligned} \phi_{B, y^*}(f) &= \phi_{B, y^*}(f - g) + \phi_{B, y^*}(g) \\ &\geq 1 - \epsilon - \|\phi_{B, y^*}\|_{\alpha^*} \|f - g\| \\ &\geq 1 - 2\epsilon \\ &> 0, \end{aligned}$$

which leads to a contradiction. Thus $\widetilde{M}_{E, \alpha}^* = \overline{\widetilde{\mathcal{M}}_{E, \alpha^*}}^{w^*}$.

□

2.6. The Maximal Dilation Norm $\|\cdot\|_\omega$

In this section, we construct the “maximal” dilation norm for the elementary dilation space M_E . Let $(\Omega, \Sigma, E, B(X, Y))$ be an operator-valued measure system. Consider the basic elements $E_{B, x} \in M_E$. It is natural to require that

$$(2.5) \quad \|E_{B, x}\| \leq \sup_{B' \in \Sigma} \|E(B \cap B')x\|.$$

Now let f be any element of M_E . If $\sum_{i=1}^N C_i E_{B_i, x_i}$ is a representation of f , then it follows from the triangle inequality that

$$\|f\| \leq \sum_{i=1}^N \sup_{B \in \Sigma} \|C_i E(B_i \cap B) x_i\|.$$

Since this holds for every representation of f , we get

$$\|f\| \leq \inf \left\{ \sum_{i=1}^N \sup_{B \in \Sigma} \|C_i E(B_i \cap B) x_i\| \right\},$$

where the infimum is taken over all representation of f . Define $\|\cdot\|_\omega : M_E \rightarrow \mathbb{R}^+ \cup \{0\}$ by

$$\|f\|_\omega = \inf \left\{ \sum_{i=1}^N \sup_{B \in \Sigma} \|C_i E(B_i \cap B) x_i\| : f = \sum_{i=1}^N C_i E_{B_i, x_i} \in M_E \right\}.$$

Then we have the following proposition.

PROPOSITION 2.42. $\|\cdot\|_\omega$ is a semi-norm on M_E and

$$\|E_{B, x}\|_\omega = \sup_{E' \in \Sigma} \|E(B \cap E') x\|$$

for every $B \in \Sigma$ and $x \in X$.

PROOF. First, we show that $\|\lambda f\|_\omega = |\lambda| \|f\|_\omega$. This is obvious when λ is zero. So suppose that $\lambda \neq 0$. If $f = \sum_{i=1}^N C_i E_{B_i, x_i}$ is a representation of f , then

$$\lambda f = \sum_{i=1}^N \lambda C_i E_{B_i, x_i} = \sum_{i=1}^N C_i E_{B_i, \lambda x_i},$$

and so we have

$$\|\lambda f\|_\omega \leq \sum_{i=1}^N \sup_{B \in \Sigma} \|C_i E(B_i \cap B)(\lambda x_i)\| = |\lambda| \sum_{i=1}^N \sup_{B \in \Sigma} \|C_i E(B_i \cap B) x_i\|.$$

Since this holds for every representation of f , it follows that $\|\lambda f\|_\omega \leq |\lambda| \|f\|_\omega$. In the same way, we have $\|f\|_\omega = \|\lambda^{-1} \lambda f\|_\omega \leq |\lambda|^{-1} \|\lambda f\|_\omega$, giving $|\lambda| \|f\|_\omega \leq \|\lambda f\|_\omega$. Therefore $\|\lambda f\|_\omega = |\lambda| \|f\|_\omega$.

Now, to prove that $\|\cdot\|_\omega$ satisfies the triangle inequality. Let $f, g \in M_E$ and let $\epsilon > 0$. It follows from the definition that we may choose representations $f = \sum_{i=1}^N C_i E_{B_i, x_i}$ and $g = \sum_{j=1}^M \tilde{C}_j E_{y_j, A_j}$ such that

$$\begin{aligned} \sum_{i=1}^N \sup_{B \in \Sigma} \|C_i E(B_i \cap B) x_i\| &\leq \|f\|_\omega + \epsilon/2, \\ \sum_{j=1}^M \sup_{A \in \Sigma} \|\tilde{C}_j E(A_j \cap A) x_j\| &\leq \|g\|_\omega + \epsilon/2. \end{aligned}$$

Then $\sum_{i=1}^N C_i E_{B_i, x_i} + \sum_{j=1}^M \tilde{C}_j E_{A_j, y_j}$ is a representation of $f + g$ and so

$$\|f + g\|_\omega \leq \sum_{i=1}^N \sup_{B \in \Sigma} \|C_i E(B_i \cap B) x_i\| + \sum_{j=1}^M \sup_{A \in \Sigma} \|\tilde{C}_j E(A_j \cap A) x_j\| \leq \|f\|_\omega + \|g\|_\omega + \epsilon.$$

Since this holds for every $\epsilon > 0$, we have $\|f + g\|_\omega \leq \|f\|_\omega + \|g\|_\omega$.

Finally, we must show that $\|E_{B,x}\|_\omega = \sup_{B' \in \Sigma} \|E(B \cap B')x\|$. On the one hand, it is clear that $\|E_{B,x}\|_\omega \leq \sup_{B' \in \Sigma} \|E(B \cap B')x\|$. On the other hand, if $E_{B,x} = \sum_{i=1}^N C_i E_{B_i, x_i}$ is a representation of $E_{B,x}$, we have

$$\begin{aligned} \sup_{B' \in \Sigma} \|E(B \cap B')x\| &= \sup_{B' \in \Sigma} \left\| \sum_{i=1}^N C_i E(B_i \cap B')x_i \right\| \\ &\leq \sup_{B' \in \Sigma} \sum_{i=1}^N \|C_i E(B_i \cap B')x_i\| \\ &\leq \sum_{i=1}^N \sup_{B' \in \Sigma} \|C_i E(B_i \cap B')x_i\|. \end{aligned}$$

Since this holds for every representation of $E_{B,x}$, it follows that $\sup_{B' \in \Sigma} \|E(B \cap B')x\| \leq \|E_{B,x}\|_\omega$. Therefore $\|E_{B,x}\|_\omega = \sup_{B' \in \Sigma} \|E(B \cap B')x\|$. \square

We define an equivalence relation R_ω on M_E by $f \sim g$ if $\|f - g\|_\omega = 0$. Then $\|\cdot\|_\omega$ is a norm on M_E . Denote by $M_{E,\omega}$ the space of the R_ω -equivalence classes of M_E endowed with the norm $\|\cdot\|_\omega$, and by $\widetilde{M}_{E,\omega}$ for its completion.

THEOREM 2.43. $\|\cdot\|_\omega$ is a dilation norm of E .

PROOF. Let $N > 0$, $\{C_i\}_{i=1}^N \subset \mathbb{C}$, $\{x_i\}_{i=1}^N \subset X$ and $\{B_i\}_{i=1}^N \subset \Sigma$. First, we show that the map $S_\omega : \widetilde{M}_{E,\omega} \rightarrow Y$ defined on $M_{E,\omega}$ by

$$S_\omega \left(\sum_{i=1}^N C_i E_{B_i, x_i} \right) = \sum_{i=1}^N E(B_i)x_i,$$

is well-defined and $\|S_\omega\| \leq 1$. If $f = \sum_{i=1}^N C_i E_{B_i, x_i}$ is a representation of f , then

$$\|S_\omega(f)\| = \left\| \sum_{i=1}^N C_i E(B_i)x_i \right\| \leq \sum_{i=1}^N \sup_{B \in \Sigma} \|C_i E(B_i \cap B)x_i\|.$$

Since this holds for every representation of f , it follows that $\|S_\omega(f)\| \leq \|f\|_\omega$. Therefore S_ω is well-defined and bounded with $\|S_\omega\| \leq 1$.

Now, to prove that the map $T_\omega : X \rightarrow \widetilde{M}_{E,\omega}$ with $T(x) = E_{\Omega, x}$ is bounded with $\|T_\omega\| \leq \|E\|$. It follows from the definition of ω that

$$\|T_\omega x\|_\omega = \|E_{\Omega, x}\|_\omega = \sup_{B \in \Sigma} \|E(B)x\| \leq \|E\| \cdot \|x\|.$$

Thus, $\|T_\omega\| \leq \|E\|$.

Finally we need to prove that the map $F_\omega : \Sigma \rightarrow B(\widetilde{M}_{E,\omega})$ defined by

$$F_\omega(B) \left(\sum_{i=1}^N C_i E_{B_i, x_i} \right) = \sum_{i=1}^N C_i E_{B \cap B_i, x_i}$$

is an operator-valued measure. By Lemma 2.25, we only need to show that F_ω is strongly countably additive and uniform bounded on $M_{E,\omega}$. The proof is very similar to that in Theorem 2.30. We include it here for completeness.

If $f = \sum_{i=1}^N C_i E_{B_i, x_i}$ is a representation of f , then

$$\begin{aligned}
\|F_\omega(B)(f)\|_\omega &= \left\| F_\omega(B) \left(\sum_{i=1}^N C_i E_{B_i, x_i} \right) \right\|_\omega \\
&= \left\| \sum_{i=1}^N C_i E_{B_i \cap B, x_i} \right\|_\omega \\
&\leq \sum_{i=1}^N \sup_{B' \in \Sigma} \|C_i E(B_i \cap B \cap B') x_i\| \\
&= \sum_{i=1}^N \sup_{B' \in \Sigma} \|C_i E(B_i \cap B') x_i\|.
\end{aligned}$$

Since this holds for every representation of f , it follows that $\|F_\omega(E)f\|_\omega \leq \|f\|_\omega$, which implies that $\|F_\omega(B)\| \leq 1$.

For the strong countably additivity of F_ω , let $\{A_j\}_{j=1}^\infty$ be a disjoint countable collection of members of Σ with union A . Then

$$\begin{aligned}
&\left\| \sum_{j=1}^M F_\omega(A_j) \left(\sum_{i=1}^N C_i E_{B_i, x_i} \right) - F_\omega(A) \left(\sum_{i=1}^N C_i E_{B_i, x_i} \right) \right\|_\omega \\
&= \left\| \sum_{j=1}^M \left(\sum_{i=1}^N C_i E_{A_j \cap B_i, x_i} \right) - \left(\sum_{i=1}^N C_i E_{A \cap B_i, x_i} \right) \right\|_\omega \\
&= \left\| \sum_{i=1}^N C_i \left(\sum_{j=1}^M E_{A_j \cap B_i, x_i} - E_{A \cap B_i, x_i} \right) \right\|_\omega \\
&= \sup_{B' \in \Sigma} \left\| \sum_{i=1}^N C_i \left(\sum_{j=1}^M E(A_j \cap B_i \cap B') x_i - E(A \cap B_i \cap B') x_i \right) \right\|_Y \\
&= \sup_{B' \in \Sigma} \left\| \sum_{i=1}^N C_i E \left(\bigcup_{j=M+1}^\infty (A_j \cap B_i \cap B') \right) x_i \right\|_Y \\
&\leq \sum_{i=1}^N |C_i| \sup_{B' \in \Sigma} \left\| E \left(\bigcup_{j=M+1}^\infty (A_j \cap B_i \cap B') \right) x_i \right\|_Y \\
&= \sum_{i=1}^N |C_i| \sup_{B' \in \Sigma} \left\| E \left(\bigcup_{j=M+1}^\infty (A_j \cap B') \right) x_i \right\|_Y \\
&= \sum_{i=1}^N |C_i| \sup_{B' \in \Sigma} \left\| \sum_{j=M+1}^\infty E(A_j \cap B') x_i \right\|_Y.
\end{aligned}$$

If $\sup_{B' \in \Sigma} \left\| \sum_{j=M+1}^\infty E(A_j \cap B') x_i \right\|_Y$ does not tend to 0 when M tends to ∞ , then we can find $\delta > 0$, a sequence of $n_1 \leq m_1 < n_2 \leq m_2 < n_3 \leq m_3 < \dots$, and

$\{B'_l\}_{l=1}^\infty \subset \Sigma$ such that

$$\left\| \sum_{j=n_l}^{m_l} E(A_j \cap B'_l) x_i \right\| \geq \delta, \quad \forall l \in \mathbb{N}.$$

Obviously, for $l \in \mathbb{N}$ and $n_l \leq j \leq m_l$, $A_j \cap B'_l$ are disjoint from each other, so

$$E \left(\bigcup_{l=1}^\infty \bigcup_{j=n_l}^{m_l} A_j \cap B'_l \right) x_i = \sum_{l=1}^\infty \sum_{j=n_l}^{m_l} E(A_j \cap B'_l) x_i,$$

which implies $\left\| \sum_{j=n_l}^{m_l} E(A_j \cap B'_l) x_i \right\| \rightarrow 0$. It is a contradiction, hence we have

$$F_\omega(A) \left(\sum_{i=1}^N C_i E_{B_i, x_i} \right) = \sum_{j=1}^\infty F_\omega(A_j) \left(\sum_{i=1}^N C_i E_{B_i, x_i} \right).$$

Thus F_ω is strongly countably additive. \square

In what follows we will refer ω as the *maximal dilation norm* of E . From Proposition 2.42 we have that both the minimal and the maximal dilation norms agree on the elementary vectors $E_{x,B}$, i.e., $\|E_{x,B}\|_\alpha = \|E_{x,B}\|_\omega$.

Finally we point out that for some special operator-valued measure systems, there are some other natural ways to construct new dilations norms. We mention two of them for which we will only give the definition but will skip the proofs.

Let $(\Omega, \Sigma, E, B(X, Y))$ be an operator-valued measure system. The *strong variation* of E is the extended nonnegative function $|E|_{\text{SOT}}$ whose value on a set $B \in \Sigma$ is given by

$$\begin{aligned} |E|_{\text{SOT}}(B) &= \sup \left\{ \sum \|E(B_i)x\| : x \in B_X, B_i\text{'s are a partition of } B \right\} \\ &= \sup_{x \in X} |E_x|(B). \end{aligned}$$

If $|E|_{\text{SOT}}(\Omega) < \infty$, then E is called an *operator-valued measure of strongly bounded variation*. Similarly, the *weak variation* of E is the extended nonnegative function $|E|_{\text{WOT}}$ whose value on a set $B \in \Sigma$ is given by

$$\begin{aligned} |E|_{\text{WOT}}(B) &= \sup \left\{ \sum |x^*(E(B_i)x)| : x \in X, y^* \in Y^*, B_i\text{'s are a partition of } B \right\} \\ &= \sup_{x \in X, y^* \in Y^*} |E_{x, y^*}|(B). \end{aligned}$$

If $|E|_{\text{WOT}}(\Omega) < \infty$, then E is called an *operator-valued measure of weakly bounded variation*.

For an operator-valued measure of strongly bounded variation (respectively, of weakly bounded variation), we have a natural approach to construct a dilation norm on M_E . Now let $f = \sum_{i=1}^N C_i E_{B_i, x_i}$ be any element of M_E . Define $\tilde{\omega}(f)$ and $\tilde{W}(f)$ by

$$\begin{aligned} \tilde{\omega}(f) &= \sup \left\{ \sum_{j=1}^M \left\| \sum_{i=1}^N C_i E(B_i \cap A_j) x_i \right\| : A_j\text{'s are a partition of } \Omega \right\} \\ &= |E_f|(\Omega). \end{aligned}$$

and

$$\begin{aligned}
\widetilde{\mathcal{W}}(f) &= \sup \left\{ \sum_{j=1}^M \left\| y^* \left(\sum_{i=1}^N C_i E(B_i \cap A_j) x_i \right) \right\| : y^* \in Y^*, A_j\text{'s are a partition of } \Omega \right\} \\
&= \sup_{y^* \in Y^*} |y^* E_f|(\Omega) \\
&= \|E_f\|(\Omega).
\end{aligned}$$

Then it can be proved that $\widetilde{\omega}$ (resp. $\widetilde{\mathcal{W}}$) is a dilation norm of E .

CHAPTER 3

Framings and Dilations

We examine the dilation theory for discrete and continuous framing-induced operator valued measures. We also provide a new self-contained proof of Naimark's Dilation Theorem based on our methods of chapter 2, because we feel that this helps clarify our approach, and also for independent interest.

3.1. Hilbertian Dilations

In the Hilbert space theory, the term “projection” is usually reserved for “orthogonal” (i.e. self-adjoint) projection. So in this Chapter we will resume this tradition. The term “idempotent” will be used to denote a not necessarily self-adjoint operator which is equal to its square. Throughout this chapter, \mathcal{H} denotes the Hilbert space and $I_{\mathcal{H}}$ is the identity operator on \mathcal{H} . Let $2^{\mathbb{N}}$ denote the family of all subsets of \mathbb{N} .

DEFINITION 3.1. Given an operator-valued measure $E : \Sigma \rightarrow B(\mathcal{H})$. Then E is called:

- (i) an operator-valued probability measure if $E(\Omega) = I_{\mathcal{H}}$,
- (ii) a (orthogonal) projection-valued measure if $E(B)$ is a (orthogonal) projection on \mathcal{H} for all $B \in \Sigma$; an idempotent-valued measure if E is a spectral in the sense of Definition 2.5,
- (iv) a positive operator-valued measure if $E(B)$ is a positive operator on \mathcal{H} for all $B \in \Sigma$.

Let $(\Omega, \Sigma, E, B(\mathcal{H}))$ be an operator-valued measure system. The definition of dilation space is the same as in section 2.2. Keep in mind that not every operator-valued measure on a Hilbert space admits a dilation space which is a Hilbert space.

DEFINITION 3.2. Let $(\Omega, \Sigma, E, B(\mathcal{H}))$ be an operator-valued measure system. We say that E has a Hilbertian dilation if it has a dilation space which is a Hilbert space. That is, E has a Hilbert dilation space if there exist a Hilbert space \mathcal{K} , two bounded linear operators $S : \mathcal{K} \rightarrow \mathcal{H}$ and $T : \mathcal{H} \rightarrow \mathcal{K}$, an idempotent-valued measure $F : \Sigma \rightarrow B(\mathcal{K})$ such that

$$E(B) = SF(B)T, \quad \forall B \in \Sigma.$$

THEOREM 3.3. *Let $(\Omega, \Sigma, E, B(\mathcal{H}))$ be an operator-valued probability measure system. If E has a Hilbert dilation space, then there exist a corresponding Hilbert dilation system $(\Omega, \Sigma, F, B(\mathcal{K}), V^*, V)$ such that $V : \mathcal{H} \rightarrow \mathcal{K}$ is an isometric embedding.*

PROOF. From Proposition 2.16, we know that if an operator-valued measure can be dilated to a spectral operator-valued measure, then it can be dilated to a spectral operator-valued probability measure. So without losing the generality, we

can assume that the corresponding dilation projection-valued measure space system $(\Omega, \Sigma, F, B(\mathcal{K}), S, T)$ is a probability measure space system, where \mathcal{K} is the Hilbert dilation space.

Since E and F are both operator-valued probability measures, we have that S is a surjection, T is an isomorphic embedding and $I_{\mathcal{H}} = ST$. Let $P = TS : \mathcal{K} \rightarrow T\mathcal{H}$. Then P is a projection from \mathcal{K} onto $T\mathcal{H}$. Hence

$$\mathcal{K} = P\mathcal{K} \oplus_{\mathcal{K}} (I_{\mathcal{K}} - P)\mathcal{K} = T\mathcal{H} \oplus_{\mathcal{K}} (I_{\mathcal{K}} - P)\mathcal{K}$$

From

$$S(I_{\mathcal{K}} - P) = S - STS = S - S = 0,$$

we get that

$$(I_{\mathcal{K}} - P)\mathcal{K} \subset \ker S.$$

On the other hand, for $z \in \mathcal{K}$, if $Sz = 0$, then

$$(I_{\mathcal{K}} - P)z = z - Pz = z - TSz = z \in (I_{\mathcal{K}} - P)\mathcal{K}$$

Thus

$$\mathcal{K} = T\mathcal{H} \oplus_{\mathcal{K}} \ker S.$$

Define

$$\tilde{\mathcal{H}} = \mathcal{H} \oplus_2 \ker S.$$

It is easy to see that the operator

$$U : \mathcal{K} = T\mathcal{H} \oplus_{\mathcal{K}} \ker S \rightarrow \tilde{\mathcal{H}} = \mathcal{H} \oplus_2 \ker S$$

defined by

$$U = T^{-1}|_{T\mathcal{H}} \oplus I_{\ker S},$$

is an isomorphic operator. Let $V = UT$ be an operator form $\mathcal{H} \rightarrow \tilde{\mathcal{H}}$. For any $x \in \mathcal{H}$, we have

$$\|Vx\|_{\tilde{\mathcal{H}}} = \|UTx\|_{\tilde{\mathcal{H}}} = \|x\|_{\mathcal{H}}.$$

Thus V is an isometric embedding.

Define

$$\tilde{F} : \Sigma \rightarrow B(\tilde{\mathcal{H}}), \quad \tilde{F}(B) = UF(B)U^{-1}.$$

Then \tilde{F} is a spectral operator-valued probability measure. Now we show that

$$E(B) = V^* \tilde{F}(B) V, \quad \forall B \in \Sigma.$$

Since

$$V^* \tilde{F}(B) V = V^* UF(B) U^{-1} V = (UT)^* UF(B) U^{-1} UT = (UT)^* UF(B) T,$$

we only need to prove that $(UT)^* U = S$, as claimed.

For any $x \in \mathcal{H}$ and $z = Tx_1 + z_2 \in \mathcal{K}$, where $z_2 \in \ker S$ and $x_1 \in \mathcal{H}$,

$$\langle (UT)^* U z, x \rangle_{\mathcal{H}} = \langle U z, UT x \rangle_{\tilde{\mathcal{H}}} = \langle UT x_1 + U z_2, x \rangle_{\tilde{\mathcal{H}}} = \langle x_1, x \rangle_{\mathcal{H}}$$

and

$$\langle Sz, x \rangle_{\mathcal{H}} = \langle ST x_1 + S z_2, x \rangle_{\mathcal{H}} = \langle x_1, x \rangle_{\mathcal{H}}$$

Hence $(UT)^* U = S$. □

The classical Naimark's Dilation Theorem tells us that if E is a positive operator-valued measure, then E has a Hilbert dilation space. Moreover, the corresponding spectral operator-valued measure F is an orthogonal projection-valued measure. Although this is a well known result in the dilation theory, for self completeness here we include a new proof which uses similar line of ideas as used in the proof of the Banach space dilation theory.

THEOREM 3.4 (Naimark's Dilation Theorem). *Let $E : \Sigma \rightarrow B(\mathcal{H})$ be a positive operator-valued measure. Then there exist a Hilbert space \mathcal{K} , a bounded linear operator $V : \mathcal{H} \rightarrow \mathcal{K}$, and an orthogonal projection-valued measure $F : \Sigma \rightarrow B(\mathcal{K})$ such that*

$$E(B) = V^* F(B) V.$$

PROOF. Let $M_E = \text{span}\{E_{B,x} : x \in \mathcal{H}, B \in \Sigma\}$ be the space induced by $(\Omega, \Sigma, E, B(\mathcal{H}))$. Now we define a sesquilinear functional $\langle \cdot, \cdot \rangle$ on this space by setting

$$\left\langle \sum_{i=1}^N E_{B_i, x_i}, \sum_{j=1}^M E_{A_j, y_j} \right\rangle = \sum_{i=1}^N \sum_{j=1}^M \langle E(B_i \cap A_j) x_i, y_j \rangle_{\mathcal{H}}.$$

Actually, if $\sum_{i=1}^N E_{B_i, x_i} = 0$, then for $1 \leq j \leq M$, $\sum_{i=1}^N E(B_i \cap A_j) x_i = 0$. Thus this sesquilinear functional is well-defined. Since for any $f = \sum_{i=1}^N E_{B_i, x_i}$ (without losing the generality, we can assume that B_i 's are disjoint from each other), we have that

$$\left\langle \sum_{i=1}^N E_{B_i, x_i}, \sum_{j=1}^N E_{B_j, x_j} \right\rangle = \sum_{i=1}^N \sum_{j=1}^N \langle E(B_i \cap B_j) x_i, x_j \rangle_{\mathcal{H}} = \sum_{i=1}^N \langle E(B_i) x_i, x_i \rangle_{\mathcal{H}} \geq 0.$$

Thus it follows that $\langle \cdot, \cdot \rangle$ is positive definite.

Obviously, this positive definite sesquilinear functional satisfies the Cauchy-Schwarz inequality,

$$|\langle f, g \rangle|^2 \leq \langle f, f \rangle \cdot \langle g, g \rangle.$$

Hence the sesquilinear functional $\langle \cdot, \cdot \rangle$ on M_E is an inner product. Let \widetilde{M}_E denote the Hilbert space that is the completion of the inner product space M_E , and the induced norm is denoted by $\|\cdot\|_{\widetilde{M}_E}$.

For every $B \in \Sigma$, define a linear map $F(B) : \widetilde{M}_E \rightarrow \widetilde{M}_E$ by

$$F(B) \left(\sum_{i=1}^N E_{B_i, x_i} \right) = \sum_{i=1}^N E_{x_i, B \cap B_i}.$$

It is easy to see that $F(B)$ is a projection. Take $\sum_{i=1}^N E_{B_i, x_i} \in \widetilde{M}_E$, then

$$\begin{aligned} \left\| F(B) \left(\sum_{i=1}^N E_{B_i, x_i} \right) \right\|_{\widetilde{M}_E} &= \sum_{i=1}^N \langle E(B \cap B_i) x_i, x_i \rangle_{\mathcal{H}} \\ &\leq \sum_{i=1}^N \langle E(B_i) x_i, x_i \rangle_{\mathcal{H}} \\ &= \left\| \sum_{i=1}^N E_{B_i, x_i} \right\|_{\widetilde{M}_E}. \end{aligned}$$

So $\|F(B)\| = 1$ or $\|F(B)\| = 0$. Moreover, we have

$$\begin{aligned} \left\langle F(B) \left(\sum_{i=1}^N E_{B_i, x_i} \right), \sum_{j=1}^M E_{A_j, y_j} \right\rangle_{\widetilde{M}_E} &= \sum_{i=1}^N \sum_{j=1}^M \langle E(B \cap B_i \cap A_j) x_i, y_j \rangle_{\mathcal{H}} \\ &= \left\langle \sum_{i=1}^N E_{B_i, x_i}, F(B) \left(\sum_{j=1}^M E_{A_j, y_j} \right) \right\rangle_{\widetilde{M}_E}. \end{aligned}$$

Thus $F(E)$ is a self-adjoint orthogonal projection.

Now we define

$$V : \mathcal{H} \rightarrow \widetilde{M}_E, \quad V(x) = E_{x, \Omega}.$$

Since

$$\|Vx\|_{\widetilde{M}_E}^2 = \langle E(\Omega)x, x \rangle_{\mathcal{H}} \leq \|E(\Omega)\| \cdot \|x\|_{\mathcal{H}}^2,$$

we know that V is bounded. Indeed, it is clear that

$$\|V\|^2 = \sup\{\langle E(\Omega)x, x \rangle_{\mathcal{H}} : \|x\|_{\mathcal{H}} \leq 1\} = \|E(\Omega)\|.$$

Observe that for any $x, y \in \mathcal{H}$,

$$\langle V^* F(B) V x, y \rangle_{\mathcal{H}} = \langle F(B)(E_{\Omega, x}), V(E_{\Omega, y}) \rangle_{\widetilde{M}_E} = \langle E(B)x, y \rangle_{\mathcal{H}},$$

so $E(B) = V^* F(B) V$, which completes the proof. \square

For general Hilbert space operator-valued measures, Don Hadwin completely characterized those that have Hilbertian dilations in terms of Hahn decompositions. Let Ω be a compact Hausdorff space and Σ be the σ -algebra of Borel subsets of Ω . Recall that a $B(\mathcal{H})$ -valued measure E is called regular if for all $x, y \in \mathcal{H}$ the complex measure given by $\langle E(\cdot)x, y \rangle$ is regular.

THEOREM 3.5. ([HA]) *Let E be a regular, bounded $B(\mathcal{H})$ -valued measure. Then the following are equivalent:*

- (i) *E has a Hahn decomposition $E = (E_1 - E_2) + i(E_3 - E_4)$, where $E_j (j = 1, 2, 3, 4)$ are positive operator-valued measures;*
- (ii) *there exist a Hilbert space \mathcal{K} , two bounded linear operators $S : \mathcal{K} \rightarrow \mathcal{H}$ and $T : \mathcal{H} \rightarrow \mathcal{K}$, an projection-valued measure $F : \Sigma \rightarrow B(\mathcal{K})$ such that*

$$E(B) = SF(B)T, \quad \forall B \in \Sigma.$$

3.2. Operator-Valued Measures Induced by Discrete Framings

Let $\{x_i\}_{i \in \mathbb{N}}$ be a non-zero frame for a separable Hilbert space \mathcal{H} . Then the mapping $E : \mathbb{N} \rightarrow B(\mathcal{H})$ defined by

$$E(B) = \sum_{i \in B} x_i \otimes x_i, \quad \forall B \in 2^{\mathbb{N}},$$

is a positive operator-valued probability measure. By Theorem 3.4, we know that E have a Hilbert dilation space \widetilde{M}_E , the corresponding operator-valued measure F is self-adjoint and idempotent. Obviously, the rank of $F(\{i\})$ is 1, so $\{E_{\{i\}, x_i} / \|x_i\|^2\}_{i \in \mathbb{N}}$ is an orthonormal basis of the Hilbert space \widetilde{M}_E . For any $B \in 2^{\mathbb{N}}$, if $i \in B$, then

$$E_{\{i\}, x}(B) = E(B \cap \{i\})x = E(\{i\})x = \langle x, x_i \rangle x_i,$$

$$\langle x, x_i \rangle \frac{E_{\{i\}, x_i}(B)}{\|x_i\|^2} = \frac{\langle x, x_i \rangle}{\|x_i\|^2} E(B \cap \{i\}) x_i = \frac{\langle x, x_i \rangle}{\|x_i\|^2} \langle x_i, x_i \rangle x_i = \langle x, x_i \rangle x_i.$$

Thus $E_{\{i\}, x} = \langle x, x_i \rangle \frac{E_{\{i\}, x_i}}{\|x_i\|^2}$. Then V is the traditional analysis operator and V^* is the traditional synthesis operator. This is because

$$Vx = E_{x, \mathbb{N}} = \sum_{i \in \mathbb{N}} \langle x, x_i \rangle \frac{E_{\{i\}, x_i}}{\|x_i\|^2}$$

and

$$V^* \left(\sum_{i \in \mathbb{N}} a_i \frac{E_{\{i\}, x_i}}{\|x_i\|^2} \right) = \sum_{i \in \mathbb{N}} a_i \frac{E(\{i\}) x_i}{\|x_i\|^2} = \sum_{i \in \mathbb{N}} a_i x_i.$$

DEFINITION 3.6. Let $\{x_i\}_{i \in \mathbb{N}}$ and $\{y_i\}_{i \in \mathbb{N}}$ are both frames for \mathcal{H} . We call $\{x_i\}_{i \in \mathbb{N}}$ and $\{y_i\}_{i \in \mathbb{N}}$ are *scale equivalent* if for each $i \in \mathbb{N}$, $y_i = \lambda_i x_i$ for some $\lambda_i \in \mathbb{C}$. $\{x_i\}_{i \in \mathbb{N}}$ and $\{y_i\}_{i \in \mathbb{N}}$ are called *unit-scale equivalent* if we can take $|\lambda_j| = 1$ for all $i \in \mathbb{N}$.

LEMMA 3.7. Two frames $\{x_i\}_{i \in \mathbb{N}}$ and $\{y_i\}_{i \in \mathbb{N}}$ are unit-scale equivalent if and only if $\{x_i\}_{i \in \mathbb{N}}$ and $\{y_i\}_{i \in \mathbb{N}}$ generate the same positive operator-valued measure.

The following theorem shows exactly when a framing induced operator-valued measure has a Hilbertian dilation.

THEOREM 3.8. Let $(x_i, y_i)_{i \in \mathbb{N}}$ be a non-zero framing for a Hilbert space \mathcal{H} . Let E be the induced operator-valued probability measure, i.e.,

$$E(B) = \sum_{i \in B} x_i \otimes y_i, \quad \forall B \subseteq \mathbb{N}.$$

Then E has a Hilbert dilation space \mathcal{K} if and only if there exist $\alpha_i, \beta_i \in \mathbb{C}, i \in \mathbb{N}$ with $\alpha_i \bar{\beta}_i = 1$ such that $\{\alpha_i x_i\}_{i \in \mathbb{N}}$ and $\{\beta_i y_i\}_{i \in \mathbb{N}}$ both are the frames for the Hilbert space \mathcal{H} .

PROOF. We first prove the sufficient condition. If E has a Hilbert dilation space \mathcal{K} , then by Theorem 2.26, there is an elementary dilation operator-valued measure system $(\Omega, \Sigma, F_{\mathcal{D}}, B(\widetilde{M}_{E, \mathcal{D}}), S_{\mathcal{D}}, T_{\mathcal{D}})$ of E . Since the norm on $M_{E, \mathcal{D}}$ is defined by

$$\left\| \sum_i E_{B_i, x_i} \right\|_{\mathcal{D}} = \left\| \sum_i F(B_i) T(x_i) \right\|_{\mathcal{K}},$$

$\widetilde{M}_{E, \mathcal{D}}$ is a Hilbert space.

Recall that the analysis operator

$$T_{\mathcal{D}} : \mathcal{H} \rightarrow \widetilde{M}_{E, \mathcal{D}}, \quad T_{\mathcal{D}}(x) = E_{x, \Omega}$$

and the synthesis operator

$$S_{\mathcal{D}} : \widetilde{M}_{E, \mathcal{D}} \rightarrow \mathcal{H}, \quad S_{\mathcal{D}} \left(\sum_i E_{B_i, x_i} \right) = \sum_i E(B_i) x_i$$

are both linear and bounded. It is easy to prove that $F_{\mathcal{D}}(\{i\})$ is rank 1 for all $i \in \mathbb{N}$. By Lemma 2.12, there is a Riesz basis ϕ_i on $\widetilde{M}_{E, \mathcal{D}}$ such that

$$F_{\mathcal{D}}(\{i\}) = \phi_i \otimes \phi_i^*.$$

Hence

$$x_i \otimes y_i = S_{\mathcal{D}}(\phi_i \otimes \phi_i^*)T_{\mathcal{D}} = (S_{\mathcal{D}}\phi_i) \otimes (T_{\mathcal{D}}^*\phi_i^*).$$

For any $x \in \mathcal{H}$, we have

$$\sum_{i \in \mathbb{N}} |\langle x, S_{\mathcal{D}}\phi_i \rangle|^2 = \sum_{i \in \mathbb{N}} |\langle S_{\mathcal{D}}^*x, z_i \rangle|^2, \quad \sum_{i \in \mathbb{N}} |\langle x, T_{\mathcal{D}}^*\phi_i^* \rangle|^2 = \sum_{i \in \mathbb{N}} |\langle T_{\mathcal{D}}x, \phi_i^* \rangle|^2.$$

Thus we know that $\{S_{\mathcal{D}}\phi_i\}_{i \in \mathbb{N}}$ and $\{T_{\mathcal{D}}^*\phi_i^*\}_{i \in \mathbb{N}}$ both are frames for \mathcal{H} and

$$x_i \otimes y_i = (S_{\mathcal{D}}\phi_i) \otimes (T_{\mathcal{D}}^*\phi_i^*).$$

So there is only a scalar adjustment $S_{\mathcal{D}}\phi_i = \alpha_i x_i$ and $T_{\mathcal{D}}^*\phi_i^* = \beta_i y_i$ with $\alpha_i \bar{\beta}_i = 1$.

For the necessary part, assume that there exist $\alpha_i, \beta_i \in \mathbb{C}, i \in \mathbb{N}$ with $\alpha_i \bar{\beta}_i = 1$ such that $\{\alpha_i x_i\}_{i \in \mathbb{N}}$ and $\{\beta_i y_i\}_{i \in \mathbb{N}}$ both are the frames for the Hilbert space \mathcal{H} . By Proposition 1.6 in [HL], we have a Hilbert space \mathcal{K} , a Riesz basis $\{z_i\}_{i \in \mathbb{N}}$ and a projection $P : \mathcal{K} \rightarrow \mathcal{H}$ such that $Pz_i = \alpha_i x_i$. Let $\{z_i^*\}_{i \in \mathbb{N}}$ be the dual Riesz basis, then $\langle z_i, z_j^* \rangle = \delta_{i,j}$. Define $S = P$

$$T : \mathcal{H} \rightarrow \mathcal{K} \quad T^*z_i^* = \beta_i y_i$$

and

$$F : 2^{\mathbb{N}} \rightarrow B(\mathcal{K}) \quad F(B) = z_i \otimes z_i^* \quad \forall B \in 2^{\mathbb{N}}.$$

Then T is a linear and bounded operator and F is a spectral operator-valued measure. For any $x \in \mathcal{H}$ and $B \in 2^{\mathbb{N}}$,

$$\begin{aligned} SF(E)Tx &= S \sum_{i \in B} \langle Tx, z_i^* \rangle z_i = \sum_{i \in B} \langle x, T^*z_i^* \rangle Pz_i \\ &= \sum_{i \in B} \langle x, \beta_i y_i \rangle \alpha_i x_i = \sum_{i \in B} \langle x, y_i \rangle x_i \\ &= E(B) \end{aligned}$$

Thus \mathcal{K} is the Hilbert dilation space of E . □

Since a completely bounded map is a linear combination of positive maps, the following Corollary immediately from Theorem 3.5 and Theorem 3.8.

COROLLARY 3.9. Let $\{x_i, y_i\}_{i \in \mathbb{N}}$ be a non-zero framing for a Hilbert space \mathcal{H} , and E be the induced operator-valued probability measure. Then E is a completely bounded map if and only if $\{x_i, y_i\}_{i \in \mathbb{N}}$ can be re-scaled to dual frames.

By using this theorem, we can prove that the first example of framing in Chapter 5 can not be scaled to a dual frame since the induced operator-valued measure doesn't have a Hilbert dilation space,

LEMMA 3.10. [Ch, Lemma 5.6.2] Assume that $\{x_i\}_{i \in \mathbb{N}}$ and $\{y_i\}_{i \in \mathbb{N}}$ are Bessel sequences in \mathcal{H} . Then the following are equivalent:

- (i) $x = \sum_{i \in \mathbb{N}} \langle x, y_i \rangle x_i, \quad \forall x \in \mathcal{H}.$
- (ii) $x = \sum_{i \in \mathbb{N}} \langle x, x_i \rangle y_i, \quad \forall x \in \mathcal{H}.$
- (iii) $\langle x, y \rangle = \sum_{i \in \mathbb{N}} \langle x, x_i \rangle \langle y_i, x_i \rangle, \quad \forall x, y \in \mathcal{H}.$

In case the equivalent conditions are satisfied, $\{x_i\}_{i \in \mathbb{N}}$ and $\{y_i\}_{i \in \mathbb{N}}$ are dual frames for \mathcal{H} .

LEMMA 3.11. [Si, Lemma 14.9] *Let $\{x_i\}_{i \in \mathbb{N}}$ be a system of vectors in \mathcal{H} . If $\sum_{i \in \mathbb{N}} x_i$ converges unconditionally in \mathcal{H} , then*

$$\sum_{i \in \mathbb{N}} \|x_i\|^2 \leq C < +\infty.$$

The following lemma can be proved by using the Orlicz-Pettis Theorem.

LEMMA 3.12. *A pair $\{x_i, y_i\}_{i \in \mathbb{N}}$ is a non-zero framing for a Hilbert space \mathcal{H} iff $\{y_i, x_i\}_{i \in \mathbb{N}}$ is a non-zero framing for \mathcal{H} .*

The following provides us a sufficient condition under which a framing induced operator-valued measure has a Hilbertian dilation.

THEOREM 3.13. *Let $\{x_i, y_i\}_{i \in \mathbb{N}}$ be a non-zero framing for a Hilbert space \mathcal{H} . If $\inf \|x_i\| \cdot \|y_i\| > 0$, then we can find $\alpha_i, \beta_i \in \mathbb{C}, i \in \mathbb{N}$ with $\alpha_i \bar{\beta}_i = 1$ such that $\{\alpha_i x_i\}_{i \in \mathbb{N}}$ and $\{\beta_i y_i\}_{i \in \mathbb{N}}$ both are frames for the Hilbert space \mathcal{H} . Hence the operator-valued measure induced by $\{x_i, y_i\}_{i \in \mathbb{N}}$ has a Hilbertian dilation.*

PROOF. Since $\{x_i, y_i\}_{i \in \mathbb{N}}$ is a non-zero framing for \mathcal{H} , we have by Lemma 3.12 that

$$x = \sum_{i \in \mathbb{N}} \langle x, y_i \rangle x_i = \sum_{i \in \mathbb{N}} \langle x, x_i \rangle y_i,$$

and the series converges unconditionally for all $x \in \mathcal{H}$. Applying Lemma 3.11 to the sequences $\{\langle x, y_i \rangle x_i\}_{i \in \mathbb{N}}$ and $\{\langle x, x_i \rangle y_i\}_{i \in \mathbb{N}}$, we get

$$\sum_{i \in \mathbb{N}} |\langle x, y_i \rangle|^2 \|x_i\|^2 < +\infty, \quad \text{and} \quad \sum_{i \in \mathbb{N}} |\langle x, x_i \rangle|^2 \|y_i\|^2 < +\infty.$$

Let

$$\alpha_i = (\|y_i\|/\|x_i\|)^{1/2} \quad \text{and} \quad \beta_i = 1/\alpha_i = (\|x_i\|/\|y_i\|)^{1/2}.$$

It is easy to see that

$$\begin{aligned} \sum_{i \in \mathbb{N}} |\langle x, \alpha_i x_i \rangle|^2 &= \sum_{i \in \mathbb{N}} \frac{|\langle x, x_i \|y_i\| \rangle|^2}{\|x_i\| \|y_i\|} \\ &\leq \frac{1}{\inf \|x_i\| \cdot \|y_i\|} \sum_{i \in \mathbb{N}} |\langle x, x_i \rangle|^2 \|y_i\|^2 < +\infty \end{aligned}$$

and

$$\begin{aligned} \sum_{i \in \mathbb{N}} |\langle x, \beta_i y_i \rangle|^2 &= \sum_{i \in \mathbb{N}} \frac{|\langle x, y_i \|x_i\| \rangle|^2}{\|x_i\| \|y_i\|} \\ &\leq \frac{1}{\inf \|x_i\| \cdot \|y_i\|} \sum_{i \in \mathbb{N}} |\langle x, y_i \rangle|^2 \|x_i\|^2 < +\infty, \end{aligned}$$

Hence $\{\alpha_i x_i\}_{i \in \mathbb{N}}$ and $\{\beta_i y_i\}_{i \in \mathbb{N}}$ are both Bessel sequence. For any $x \in \mathcal{H}$, we have

$$x = \sum_{i \in \mathbb{N}} \langle x, \beta_i y_i \rangle \alpha_i x_i = \sum_{i \in \mathbb{N}} \langle x, \alpha_i x_i \rangle \beta_i y_i.$$

By Lemma 3.11, we conclude that $\{\alpha_i x_i\}_{i \in \mathbb{N}}$ and $\{\beta_i y_i\}_{i \in \mathbb{N}}$ both are frames for \mathcal{H} . \square

Recall from Dai and Larson [DL] that a *unitary system* \mathcal{U} is a subset of the unitary operators acting on a separable Hilbert space \mathcal{H} which contains the identity operator I . We have the following consequence immediately follows from Theorem 3.13:

COROLLARY 3.14. Let \mathcal{U}_1 and \mathcal{U}_2 be unitary systems on a separable Hilbert space \mathcal{H} . If there exist $x, y \in \mathcal{H}$ such that $\{\mathcal{U}_1 x, \mathcal{U}_2 y\}$ is a framing of \mathcal{H} , then $\{\mathcal{U}_1 x\}$ and $\{\mathcal{U}_2 y\}$ both are frames for \mathcal{H} .

We remark that there exist sequences $\{x_n\}$ and $\{y_n\}$ in a Hilbert space \mathcal{H} with the following properties:

(i) $x = \sum_n \langle x, x_n \rangle y_n$ hold for all x in a dense subset of \mathcal{H} , and the convergence is unconditionally.

(ii) $\inf \|x_n\| \cdot \|y_n\| > 0$.

(iii) $\{x_n, y_n\}$ is not a framing.

Here is a simple example of this type: Let $\mathcal{H} = L^2[0, 1]$, and $g(t) = t^{1/4}$, $f(t) = 1/g(t)$. Define $x_n(t) = e^{2\pi i n t} f(t)$ and $y_n(t) = e^{2\pi i n t} g(t)$. Then it is easy to verify (i) and (ii). For (iii), we consider the convergence of the series

$$\sum_{n \in \mathbb{Z}} \langle f, x_n \rangle y_n.$$

Note that $\|\langle f, x_n \rangle y_n\|^2 = |\langle f, x_n \rangle|^2 \cdot \|g\|^2$ and $\{\langle f, x_n \rangle\}$ is not in ℓ^2 (since $f^2 \notin L^2[0, 1]$). Thus, from Lemma 3.11, we have that $\sum_n \langle f, x_n \rangle y_n$ can not be convergent unconditionally. Therefore $\{x_n, y_n\}$ is not a framing.

3.3. Operator-Valued Measures Induced by Continuous Frames

We mainly examine the positive-operator valued measure induced by continuous frames investigated by Gabor and Han in [GH] and also by Fornasier and Rauhut in [FR]. At the same time, we will introduce a few new “frames” including operator-valued continuous frames and operator-valued μ -frames, both generalize the concept of continuous frames.

DEFINITION 3.15. Let \mathcal{H} be a separable Hilbert space and Ω be a σ -locally compact (σ -compact and locally compact) Hausdorff space endowed with a positive Radon measure μ with $\text{supp } \mu = \Omega$. A weakly continuous function $\mathcal{F} : \Omega \rightarrow \mathcal{H}$ is called a *continuous frame* if there exist constants $0 < C_1 \leq C_2 < \infty$ such that

$$C_1 \|x\|^2 \leq \int_{\Omega} |\langle x, \mathcal{F}(\omega) \rangle|^2 d\mu(\omega) \leq C_2 \|x\|^2, \quad \forall x \in \mathcal{H}.$$

If $C_1 = C_2$, then the frame is called *tight*. Associated to \mathcal{F} is the frame operator $S_{\mathcal{F}}$ defined by

$$S_{\mathcal{F}} : \mathcal{H} \rightarrow \mathcal{H}, \quad \langle S_{\mathcal{F}}(x), y \rangle := \int_{\Omega} \langle x, \mathcal{F}(\omega) \rangle \cdot \langle \mathcal{F}(\omega), y \rangle d\mu(\omega).$$

Then $S_{\mathcal{F}}$ is a bounded, positive, and invertible operator. We define the following transform associated to \mathcal{F} ,

$$V_{\mathcal{F}} : \mathcal{H} \rightarrow L^2(\Omega, \mu), \quad V_{\mathcal{F}}(x)(\omega) := \langle x, \mathcal{F}(\omega) \rangle.$$

Its adjoint operator is given by

$$V_{\mathcal{F}}^* : L^2(\Omega, \mu) \rightarrow \mathcal{H}, \quad \langle V_{\mathcal{F}}^*(f), x \rangle := \int_{\Omega} f(\omega) \langle \mathcal{F}(\omega), x \rangle d\mu(\omega).$$

A weakly continuous function $\mathcal{F} : \Omega \rightarrow \mathcal{H}$ is called *Bessel* if there exist a positive constant C such that

$$\int_{\Omega} |\langle x, \mathcal{F}(\omega) \rangle|^2 d\mu(\omega) \leq C \|x\|^2, \quad \forall x \in \mathcal{H}.$$

Let \mathcal{B} be the Borel algebra of Ω and $\mathcal{F} : \Omega \rightarrow \mathcal{H}$ be Bessel. Then the mapping

$$E_{\mathcal{F}} : \mathcal{B} \rightarrow B(\mathcal{H}), \quad \langle E_{\mathcal{F}}(B)x, y \rangle = \int_B \langle x, \mathcal{F}(\omega) \rangle \cdot \langle \mathcal{F}(\omega), y \rangle d\mu(\omega)$$

is well-defined since we have

$$\begin{aligned} |\langle E_{\mathcal{F}}(B)x, y \rangle| &= \left| \int_B \langle x, \mathcal{F}(\omega) \rangle \cdot \langle \mathcal{F}(\omega), y \rangle d\mu(\omega) \right| \\ &\leq \int_{\Omega} |\langle x, \mathcal{F}(\omega) \rangle| \cdot |\langle \mathcal{F}(\omega), y \rangle| d\mu(\omega) \\ &\leq \left(\int_{\Omega} |\langle x, \mathcal{F}(\omega) \rangle|^2 d\mu(\omega) \right)^{1/2} \cdot \left(\int_{\Omega} |\langle \mathcal{F}(\omega), y \rangle|^2 d\mu(\omega) \right)^{1/2} \\ &\leq C \cdot \|x\| \cdot \|y\|. \end{aligned}$$

and therefore $\|E_{\mathcal{F}}(B)\| \leq C$ for all $E \in \mathcal{B}$.

LEMMA 3.16. $E_{\mathcal{F}}$ defined above is a positive operator-valued measure.

PROOF. The positivity follows from that

$$\langle E_{\mathcal{F}}(B)x, x \rangle = \int_{\Omega} |\langle x, \mathcal{F}(\omega) \rangle|^2 d\mu(\omega) \geq 0$$

holds for all x in \mathcal{H} .

Assume that $\{B_i\}_{i \in \mathbb{N}}$ is a sequence of disjoint Borel sets of Ω with union B . Then we have

$$\begin{aligned} \sum_{i \in \mathbb{N}} \langle E_{\mathcal{F}}(B_i)x, y \rangle &= \sum_{i \in \mathbb{N}} \int_{B_i} \langle x, \mathcal{F}(\omega) \rangle \cdot \langle \mathcal{F}(\omega), y \rangle d\mu(\omega) \\ &= \int_B \langle x, \mathcal{F}(\omega) \rangle \cdot \langle \mathcal{F}(\omega), y \rangle d\mu(\omega) \\ &= \langle E_{\mathcal{F}}(B)x, y \rangle \end{aligned}$$

for all x, y in \mathcal{H} . Thus $E_{\mathcal{F}}$ weakly countably additive, and hence it is a positive operator-valued measure. \square

LEMMA 3.17. Let \mathcal{F}_1 and \mathcal{F}_2 be two Bessel functions from Ω to \mathcal{H} . Then \mathcal{F}_1 and \mathcal{F}_2 induce the same positive operator-valued measure if and only if $\mathcal{F}_1 = u \cdot \mathcal{F}_2$ where u is a unimodular function on Ω .

PROOF. Since $E_{\mathcal{F}_1} = E_{\mathcal{F}_2}$, we have

$$\int_E |\langle x, \mathcal{F}_1(\omega) \rangle|^2 d\mu(\omega) = \langle E_{\mathcal{F}_1}(B)x, x \rangle = \langle E_{\mathcal{F}_2}(B)x, x \rangle = \int_B |\langle x, \mathcal{F}_2(\omega) \rangle|^2 d\mu(\omega)$$

for all $B \in \Sigma$ and $x \in \mathcal{H}$. Then by the continuity, we have $|\langle x, \mathcal{F}_1(\omega) \rangle| = |\langle x, \mathcal{F}_2(\omega) \rangle|$ for all $x \in \mathcal{H}$ and $\omega \in \Omega$. Thus we have

$$|\langle \mathcal{F}_1(\omega), \mathcal{F}_2(\omega) \rangle| = \|\mathcal{F}_1(\omega)\|^2 = \|\mathcal{F}_2(\omega)\|^2 = \|\mathcal{F}_1(\omega)\| \cdot \|\mathcal{F}_2(\omega)\|.$$

Then there is a unimodular function on Ω such that $\mathcal{F}_1(\omega) = u(\omega) \cdot \mathcal{F}_2(\omega)$. \square

From Theorem 3.4, we know that $E_{\mathcal{F}}$ has a Hilbert dilation space $\widetilde{M}_{E_{\mathcal{F}}}$. The following theorem shows that $\widetilde{M}_{E_{\mathcal{F}}}$ is isometric to $L^2(\Omega, \mu)$.

THEOREM 3.18. *Suppose that $\text{supp } \mathcal{F} = \Omega$. Then the dilation Hilbert space $\widetilde{M}_{E_{\mathcal{F}}}$ is isometric to $L^2(\Omega, \mu)$ under the following natural linear surjective isometry*

$$U : \widetilde{M}_{E_{\mathcal{F}}} \rightarrow L^2(\Omega, \mu), \quad U(E_{\mathcal{F}B,x})(\omega) = \chi_B(\omega) \cdot \langle x, \mathcal{F}(\omega) \rangle$$

PROOF. Without losing the generality, we can assume that $\{B_i\}$ is a disjoint countable collection of members of Σ . Then we have

$$\begin{aligned} \left\| \sum_i E_{\mathcal{F}B_i, x_i} \right\|_{\widetilde{M}_{E_{\mathcal{F}}}}^2 &= \sum_i \langle E_{\mathcal{F}}(B_i) x_i, x_i \rangle \\ &= \sum_i \int_{B_i} |\langle x_i, \mathcal{F}(\omega) \rangle|^2 d\mu(\omega) \\ &= \int_{\Omega} \left| \sum_i \chi_{B_i}(\omega) \cdot \langle x_i, \mathcal{F}(\omega) \rangle \right|^2 d\mu(\omega) \\ &= \left\| U \left(\sum_i E_{\mathcal{F}B_i, x_i} \right) \right\|_{L^2(\Omega, \mu)}^2. \end{aligned}$$

Thus U is an isometry.

Now we prove that U is surjective. Because all the functions $\chi_B(\omega)$ are dense in $L^2(\Omega, \mu)$, where B is any set in Ω . So we only need to approximate $\chi_B(\omega)$ with $0 < \mu(B) < \infty$. Since μ is a Radon measure which is inner regular, without losing the generality, we can assume that B is compact. Then for any $\omega \in E$, we can find an $x_0 \in \mathcal{H}$ such that $\langle x_0, \mathcal{F}(\omega) \rangle \neq 0$. Let $x_{\omega} = x_0 / \langle x_0, \mathcal{F}(\omega) \rangle$. Then

$$\langle x_{\omega}, \mathcal{F}(\omega) \rangle = 1.$$

Since \mathcal{F} is a weakly continuous function from Ω to \mathcal{H} , there is a neighborhood \mathcal{U}_{ω} of ω such that for all $v \in \mathcal{U}_{\omega}$,

$$|\langle x_{\omega}, \mathcal{F}(v) \rangle - 1| \leq \sqrt{\frac{\epsilon}{\mu(B)}}.$$

Then we have $E \subset \cup_{\omega \in B} \mathcal{U}_{\omega}$. Since E is compact, there are finitely many $\{\omega_i\}_{1 \leq i \leq N}$ such that $B \subset \cup_{1 \leq i \leq N} \mathcal{U}_{\omega_i}$, and hence $B = \cup_{1 \leq i \leq N} (\mathcal{U}_{\omega_i} \cap B)$. We can find a sequence of disjoint subsets $\widetilde{\mathcal{U}}_{\omega_i}$ of $\mathcal{U}_{\omega_i} \cap B$ such that $B = \cup_{1 \leq i \leq N} \widetilde{\mathcal{U}}_{\omega_i}$. Then we have

$$\begin{aligned} &\left\| U \left(\sum_{i=1}^N E_{\mathcal{F}x_{\omega_i}, \widetilde{\mathcal{U}}_{\omega_i}} \right) - \chi_B(\omega) y \right\|_{L^2(\Omega, \mu)}^2 \\ &= \int_{\Omega} \left| \sum_{i=1}^N \chi_{\widetilde{\mathcal{U}}_{\omega_i}}(\omega) \langle x_{\omega_i}, \mathcal{F}(\omega) \rangle - \chi_B(\omega) \right|^2 d\mu(\omega) \\ &= \sum_{i=1}^N \int_{\widetilde{\mathcal{U}}_{\omega_i}} |\langle x_{\omega_i}, \mathcal{F}(\omega) \rangle - 1|^2 d\mu(\omega) \\ &\leq \sum_{i=1}^N \mu(\widetilde{\mathcal{U}}_{\omega_i}) \frac{\epsilon}{\mu(B)} = \epsilon. \end{aligned}$$

Therefore U is surjective as claimed. \square

Then we can write the orthogonal projection-valued measure F on the dilation space as follows:

THEOREM 3.19. *Let $F(B)$ be the orthogonal projection-valued measure in Theorem 3.4. Then we have $F(B) = U^* \chi_B U$ for every $B \in \Sigma$.*

PROOF. From Theorem 3.4, we know that

$$F(B) \left(\sum_i E_{\mathcal{F}B_i, x_i} \right) = \sum_i E_{\mathcal{F}B \cap B_i, x_i}.$$

Also for any $y \in \mathcal{H}$ and $A \in \Sigma$, we have

$$\begin{aligned} \langle U^* (\chi_B(\omega) \langle x, \mathcal{F}(\omega) \rangle), E_{\mathcal{F}A, y} \rangle_{\widetilde{M}_{E_{\mathcal{F}}}} &= \langle \chi_B(\omega) \langle x, \mathcal{F}(\omega) \rangle, U(E_{\mathcal{F}A, y}) \rangle_{L^2(\Omega, \mu)} \\ &= \langle \chi_B(\omega) \langle x, \mathcal{F}(\omega) \rangle, \chi_A(\omega) \langle y, \mathcal{F}(\omega) \rangle \rangle_{L^2(\Omega, \mu)} \\ &= \int_{B \cap A} \langle x, \mathcal{F}(\omega) \rangle \cdot \langle \mathcal{F}(\omega), y \rangle d\mu(\omega), \end{aligned}$$

and

$$\begin{aligned} \langle E_{\mathcal{F}B, x}, E_{\mathcal{F}A, y} \rangle_{\widetilde{M}_{E_{\mathcal{F}}}} &= \langle E_{\mathcal{F}(B \cap A)} x, y \rangle_{\mathcal{H}} \\ &= \int_{B \cap A} \langle x, \mathcal{F}(\omega) \rangle \cdot \langle \mathcal{F}(\omega), y \rangle d\mu(\omega). \end{aligned}$$

Thus

$$U^* (\chi_B(\omega) \langle x, \mathcal{F}(\omega) \rangle) = E_{\mathcal{F}B, x}.$$

Since

$$\begin{aligned} U^* \chi_B U \left(\sum_i E_{\mathcal{F}B_i, x_i} \right) &= \sum_i U^* \chi_B(\omega) (U E_{\mathcal{F}B_i, x_i}) \\ &= \sum_i U^* \chi_B(\omega) \chi_{B_i}(\omega) \langle x_i, \mathcal{F}(\omega) \rangle \\ &= \sum_i U^* \chi_{B \cap B_i}(\omega) \langle x_i, \mathcal{F}(\omega) \rangle \\ &= \sum_i E_{\mathcal{F}B \cap B_i, x_i} \\ &= F(B) \left(\sum_i E_{\mathcal{F}B_i, x_i} \right), \end{aligned}$$

we get $F(B) = U^* \chi_B U$, as claimed. \square

THEOREM 3.20. *Suppose that $\text{supp } \mathcal{F} = \Omega$. If*

$$L := \inf \{ \|E_{\mathcal{F}}(B)\| : \|E_{\mathcal{F}}(B)\| > 0 \} > 0,$$

then Ω is at most countable, that is, every point in Ω is an open set.

PROOF. Let $L = \inf\{\|E_{\mathcal{F}}(B)\| : \|E_{\mathcal{F}}(B)\| > 0\} > 0$. First we show that for any open subset U of Ω , we have $\|E_{\mathcal{F}}(U)\| > 0$. Choose $\omega_U \in U$ and $x_U \in \mathcal{H}$ such that $\langle x_U, \mathcal{F}(\omega_U) \rangle \neq 0$. Then we have

$$\begin{aligned} \langle E_{\mathcal{F}}(U)x_U, x_U \rangle &= \int_U |\langle x_U, \mathcal{F}(\omega) \rangle|^2 d\mu(\omega) \\ &\geq \int_{\{\omega \in U : |\langle x_U, \mathcal{F}(\omega) \rangle| > |\langle x_U, \mathcal{F}(\omega_U) \rangle|/2\}} |\langle x_U, \mathcal{F}(\omega) \rangle|^2 d\mu(\omega) \\ &\geq \mu\{\omega \in U : |\langle x_U, \mathcal{F}(\omega) \rangle| > |\langle x_U, \mathcal{F}(\omega_U) \rangle|/2\} \cdot |\langle x_U, \mathcal{F}(\omega_U) \rangle|^2/4 \\ &> 0. \end{aligned}$$

Thus $\|E_{\mathcal{F}}(U)\| > 0$, and so $\|E_{\mathcal{F}}(U)\| \geq L$ for any open set U .

Fix any $\omega_0 \in \Omega$. Since Ω is locally compact, we can choose a compact neighborhood U_{ω_0} of ω_0 . Since $\mathcal{F}(U_{\omega_0})$ is weakly compact, we have

$$M := \sup_{w \in U_{\omega_0}} \|\mathcal{F}(w)\| < \infty.$$

Thus, for any open subset $U \subset U_{\omega_0}$, we get

$$\begin{aligned} |\langle E_{\mathcal{F}}(U)x, y \rangle| &= \left| \int_U \langle x, \mathcal{F}(\omega) \rangle \cdot \langle \mathcal{F}(\omega), y \rangle d\mu(\omega) \right| \\ &\leq \int_U |\langle x, \mathcal{F}(\omega) \rangle| \cdot |\langle \mathcal{F}(\omega), y \rangle| d\mu(\omega) \\ &\leq \left(\int_U |\langle x, \mathcal{F}(\omega) \rangle|^2 d\mu(\omega) \right)^{1/2} \cdot \left(\int_U |\langle \mathcal{F}(\omega), y \rangle|^2 d\mu(\omega) \right)^{1/2} \\ &\leq \left(\int_U \|x\|^2 \cdot \|\mathcal{F}(\omega)\|^2 d\mu(\omega) \right)^{1/2} \cdot \left(\int_U \|\mathcal{F}(\omega)\|^2 \cdot \|y\|^2 d\mu(\omega) \right)^{1/2} \\ &\leq \left(\int_U \|x\|^2 \cdot M^2 d\mu(\omega) \right)^{1/2} \cdot \left(\int_U M^2 \cdot \|y\|^2 d\mu(\omega) \right)^{1/2} \\ &\leq \mu(U) \cdot M^2 \cdot \|x\| \cdot \|y\|. \end{aligned}$$

This implies that $\|E_{\mathcal{F}}(U)\| \leq M^2 \mu(U)$. By $\|E_{\mathcal{F}}(U)\| \geq L$, we obtain that $\mu(U) \geq L/M^2 > 0$ for all open subsets of U_{ω_0} . Let

$$\mathcal{V}_{\omega_0} = \{V : V \text{ is an open subset of } U_{\omega_0} \text{ containing } \omega_0\}$$

and

$$\tilde{L} := \inf_{V \in \mathcal{V}_{\omega_0}} \mu(V) \geq L/M^2 > 0.$$

We can choose $V_{\omega_0} \in \mathcal{V}_{\omega_0}$ such that $\mu(V_{\omega_0}) < \tilde{L} + L/(2M^2)$.

Now, we prove that $V_{\omega_0} = \{\omega_0\}$. If there is another $\omega_1 \in V_{\omega_0}$ and $\omega_1 \neq \omega_0$, since Ω is Hausdorff, we can find an open set $W_{\omega_0} \subset V_{\omega_0} \subset U_{\omega_0}$ containing ω_0 , and an open set $W_{\omega_1} \subset V_{\omega_0} \subset U_{\omega_0}$ containing ω_1 such that $W_{\omega_0} \cap W_{\omega_1} = \emptyset$. Since $W_{\omega_0} \in \mathcal{V}_{\omega_0}$, we have $\mu(W_{\omega_0}) \geq \tilde{L}$. Thus,

$$\tilde{L} + L/(2M^2) > \mu(V_{\omega_0}) \geq \mu(W_{\omega_0}) + \mu(W_{\omega_1}) \geq \tilde{L} + L/M^2,$$

which is a contradiction. So $\{\omega_0\} = V_{\omega_0}$ is an open set, hence Ω is at most countable. \square

COROLLARY 3.21. If \mathcal{H} is separable and $E_{\mathcal{F}}$ is a projection-valued measure, then Ω is countable.

We briefly discuss one generalization of a continuous frame.

DEFINITION 3.22. A function $\mathcal{F} : \Omega \rightarrow B(\mathcal{H}, \mathcal{H}_0)$ is called an *operator-valued μ -frame* if it is weakly Bochner measurable and if there exist two constants $A, B > 0$ such that

$$A\|x\|_{\mathcal{H}}^2 \leq \int_{\Omega} \|\mathcal{F}(\omega)x\|_{\mathcal{H}_0}^2 d\mu(\omega) \leq B\|x\|_{\mathcal{H}}^2$$

holds for all $x \in \mathcal{H}$.

Similar to the continuous frame case we have:

THEOREM 3.23. *Then the mapping*

$$\varphi_{\mathcal{F}} : \Sigma \rightarrow B(\mathcal{H}) \quad \langle E_{\mathcal{F}}(B)x, y \rangle_{\mathcal{H}} = \int_B \langle \mathcal{F}(\omega)x, \mathcal{F}(\omega)y \rangle_{\mathcal{H}_0} d\mu(\omega);$$

is an operator-valued measure.

Define an operator $\theta_{\mathcal{F}} : \mathcal{H} \rightarrow L^2(\mu; \mathcal{H}_0)$ by

$$(\theta_{\mathcal{F}}x)(\omega) = \mathcal{F}(\omega)x, \quad \forall x \in \mathcal{H}, \omega \in \Omega.$$

It is easy to see that $\theta_{\mathcal{F}}$ is a bounded linear operator. In fact, it is injective and bounded below. For operator-valued μ -frames the dilation space is very much similar to the regular frames case.

THEOREM 3.24. *$L^2(\mu; \mathcal{H}_0)$ is a Hilbert dilation space of $E_{\mathcal{F}}$.*

PROOF. Define a mapping $F_{\mathcal{F}} : \Sigma \rightarrow B(L^2(\mu; \mathcal{H}_0))$ by

$$(F_{\mathcal{F}}(B)f)(\omega) = \begin{cases} f(\omega), & \omega \in B, \\ 0, & \omega \notin B. \end{cases}$$

Then $F_{\mathcal{F}}$ is a self-adjoint projection-valued measure.

Since

$$\begin{aligned} \langle \theta_{\mathcal{F}}^* F_{\mathcal{F}}(B) \theta_{\mathcal{F}} x, y \rangle_{\mathcal{H}} &= \langle F_{\mathcal{F}}(B) \theta_{\mathcal{F}} x, \theta_{\mathcal{F}} y \rangle_{L^2(\mu; \mathcal{H}_0)} \\ &= \langle F_{\mathcal{F}}(B) \mathcal{F}(\omega)x, \mathcal{F}(\omega)y \rangle_{L^2(\mu; \mathcal{H}_0)} \\ &= \langle \chi_E(\omega) \mathcal{F}(\omega)x, \mathcal{F}(\omega)y \rangle_{L^2(\mu; \mathcal{H}_0)} \\ &= \int_B \langle \mathcal{F}(\omega)x, \mathcal{F}(\omega)x \rangle_{\mathcal{H}_0} d\mu(\omega) \\ &= \langle E_{\mathcal{F}}(B)x, y \rangle_{\mathcal{H}} \end{aligned}$$

Thus $L^2(\mu; \mathcal{H}_0)$ is a Hilbert dilation space of $E_{\mathcal{F}}$. □

The operator $\theta_{\mathcal{F}}$ is the usual analysis operator and $\theta_{\mathcal{F}}^*$ is the usual synthesis operator. The frame operator on \mathcal{H} is defined by

$$S_{\mathcal{F}} : \mathcal{H} \rightarrow \mathcal{H}, \quad S_{\mathcal{F}} = \theta_{\mathcal{F}}^* \theta_{\mathcal{F}}.$$

For any $x, y \in \mathcal{H}$, we have

$$\begin{aligned}\langle S_{\mathcal{F}}x, y \rangle_{\mathcal{H}} &= \langle \theta_{\mathcal{F}}x, \theta_{\mathcal{F}}y \rangle_{L^2(\mu; \mathcal{H}_0)} \\ &= \int_{\Omega} \langle \mathcal{F}(\omega)x, \mathcal{F}(\omega)y \rangle_{\mathcal{H}_0} d\mu(\omega) \\ &= \int_{\Omega} \langle \mathcal{F}^*(\omega)\mathcal{F}(\omega)x, y \rangle_{\mathcal{H}} d\mu(\omega)\end{aligned}$$

Thus

$$\begin{aligned}S_{\mathcal{F}}x &= \int_{\Omega} \mathcal{F}^*(\omega)\mathcal{F}(\omega)x d\mu(\omega). \\ \theta_{\mathcal{F}}^*f &= \int_{\Omega} \mathcal{F}^*(\omega)f(\omega) d\mu(\omega),\end{aligned}$$

where the equation is in weak sense, i.e.

$$\langle \theta_{\mathcal{F}}^*f, x \rangle_{\mathcal{H}} = \int_{\Omega} \langle \mathcal{F}^*(\omega)f(\omega), x \rangle_{\mathcal{H}} d\mu(\omega),$$

and

$$\langle S_{\mathcal{F}}x, y \rangle_{\mathcal{H}} = \int_{\Omega} \langle \mathcal{F}^*(\omega)\mathcal{F}(\omega)x, y \rangle_{\mathcal{H}} d\mu(\omega).$$

Therefore for each operator-valued μ -frame, we have four associated operator-valued measures on (Ω, Σ) , namely:

- (i) The frame operator-valued measure $E_{\mathcal{F}} : \Sigma \rightarrow B(\mathcal{H})$ is defined by

$$\langle E_{\mathcal{F}}(B)x, y \rangle_{\mathcal{H}} = \int_B \langle \mathcal{F}(\omega)x, \mathcal{F}(\omega)y \rangle_{\mathcal{H}_0} d\mu(\omega);$$

- (ii) The analysis operator-valued measure $\alpha_{\mathcal{F}} : \Sigma \rightarrow B(\mathcal{H}, L^2(\mu; \mathcal{H}_0))$ is defined by

$$(\alpha_{\mathcal{F}}(B)x)(\omega) = \begin{cases} \mathcal{F}(\omega)x, & \omega \in B, \\ 0, & \omega \notin B; \end{cases}$$

- (iii) The synthesis operator-valued measure $\sigma_{\mathcal{F}} : \Sigma \rightarrow B(L^2(\mu; \mathcal{H}_0), \mathcal{H})$ is defined by

$$\langle \sigma_{\mathcal{F}}(B)f, x \rangle_{\mathcal{H}} = \int_B \langle f(\omega), \mathcal{F}(\omega)x \rangle_{\mathcal{H}_0} d\mu(\omega);$$

- (iv) The self-adjoint projection-valued measure $F_{\mathcal{F}} : \Sigma \rightarrow B(L^2(\mu; \mathcal{H}_0))$ is defined by

$$(F_{\mathcal{F}}(E)f)(\omega) = \begin{cases} f(\omega), & \omega \in E, \\ 0, & \omega \notin E. \end{cases}$$

CHAPTER 4

Dilations of Maps

In this chapter we establish some dilation results for general linear mappings of algebras, mainly focusing on (not necessarily cb-maps) on von Neumann algebras, and more generally on Banach algebras. The ideas for our proofs come indirectly from our methods in Chapter 2 for OVM's.

We begin with a possibly known purely algebraic result (Proposition 4.1) which shows that dilations of linear maps are always possible even in the absence of any topological structure. In the presence of additional hypotheses stronger results are possible: When a domain algebra, mapping and range space have strong continuity and/or structural properties we seek similar properties for the dilation. This plan led to our other results. Theorem 4.10 states that for any Banach algebra \mathcal{A} and any bounded linear operator ϕ from \mathcal{A} to $B(\mathcal{H})$ on a Banach space \mathcal{H} , there exist a Banach space Z , a bounded linear unital homomorphism $\pi : \mathcal{A} \rightarrow B(Z)$, and bounded linear operators $T : \mathcal{H} \rightarrow Z$ and $S : Z \rightarrow \mathcal{H}$ such that

$$\phi(a) = S\pi(a)T$$

for all $a \in \mathcal{A}$. In the case that the Banach algebra is an abelian purely atomic von Neumann algebra \mathcal{A} and \mathcal{H} is a Hilbert space, and ϕ is normal (i.e. ultraweakly continuous), then there is a normal dilation π (Theorem 4.7). If ϕ is not cb then the dilation space cannot be a Hilbert space; the normality of the dilation is with respect to the natural ultraweak topology on $B(Z)$. It is not known the extent to which this result can be generalized (i.e. achieving normality of the dilation).

4.1. Algebraic Dilations

PROPOSITION 4.1. If A is unital algebra, V a vector space, and $\phi : A \rightarrow L(V)$ a linear map, then there exists a vector space W , a unital homomorphism $\pi : A \rightarrow L(W)$, and linear maps $T : V \rightarrow W$, $S : W \rightarrow V$, such that

$$\phi(\cdot) = S\pi(\cdot)T.$$

PROOF. For $a \in A, x \in V$, define $\alpha_{a,x} \in L(A, V)$ by

$$\alpha_{a,x}(\cdot) := \phi(\cdot a)x.$$

Let $W := \text{span}\{\alpha_{a,x} : a \in A, x \in V\} \subset L(A, V)$. Define $\pi : A \rightarrow L(W)$ by $\pi(a)(\alpha_{b,x}) := \alpha_{ab,x}$. It is easy to see that π is a unital homomorphism. For $x \in V$ define $T : V \rightarrow L(A, V)$ by $T_x := \alpha_{I,x} = \phi(\cdot)x = \phi(\cdot)x$. Define $S : W \rightarrow W$ by setting $S(\alpha_{a,x}) := \phi(a)x$ and extending linearly to W . If $a \in A, x \in V$ are arbitrary, we have $S\pi(a)Tx = S\pi(a)\alpha_{I,x} = S\alpha_{a,x} = \phi(a)x$. Hence $\phi = S\pi T$. \square

4.2. The Commutative Case

We first examine a mapping from the commutative C^* -algebra ℓ_∞ into $B(\mathcal{H})$ which is induced by a framing on a Hilbert space \mathcal{H} . Here ℓ_∞ means with respect to a countable or finite index set J , and it is well known that every separably acting purely atomic abelian von Neumann algebra is equivalent to some ℓ_∞ via an ultraweakly continuous $*$ -isomorphism.

THEOREM 4.2. *Let \mathcal{H} be a separable Hilbert space and let (x_i, y_i) be a framing of \mathcal{H} . Then the mapping ϕ from ℓ_∞ into $B(\mathcal{H})$ defined by*

$$\phi : \ell_\infty \rightarrow B(\mathcal{H}), \quad (a_i) \rightarrow \sum^{SOT} a_i x_i \otimes y_i$$

is well-defined, unital, linear and ultraweakly continuous.

PROOF. Since (x_i, y_i) is a framing of \mathcal{H} , for any $x \in \mathcal{H}$,

$$x = \sum_i \langle x, y_i \rangle x_i$$

converges unconditionally.

For any $(a_i) \in c_{00}$, define a bounded operator $U_{(a_i)}$ as follows

$$U_{(a_i)} : \mathcal{H} \rightarrow \mathcal{H}, \quad U_{(a_i)}(x) = \sum a_i \langle x, y_i \rangle x_i.$$

For any $x \in \mathcal{H}$, since $\sum_i \langle x, y_i \rangle x_i$ converges unconditionally, by [DJT] Theorem 1.9, we know that

$$\sup_{(b_i) \in B_1(\ell_\infty)} \left\| \sum b_i \langle x, y_i \rangle x_i \right\| < +\infty.$$

Thus,

$$\sup_{(a_i) \in B_1(c_{00})} \|U_{(a_i)}(x)\| \leq \sup_{(b_i) \in B_1(\ell_\infty)} \left\| \sum b_i \langle x, y_i \rangle x_i \right\| < +\infty.$$

Then by the Uniform Boundedness Principle,

$$\sup_{(a_i) \in B_1(c_{00})} \|U_{(a_i)}\| < +\infty.$$

It follows that

$$K_u = \sup_{x \in B_1(\mathcal{H})} \sup_{(\sigma_i) \subset \mathbb{D}} \left\| \sum \sigma_i \langle x, y_i \rangle x_i \right\| < +\infty.$$

Thus, for all $(a_i) \in \ell_\infty$ and $x \in \mathcal{H}$,

$$\left\| \sum a_i \langle x, y_i \rangle x_i \right\| \leq K_u \|a_i\|_{\ell_\infty} \|x\|.$$

Hence F is well-defined, unital, linear and bounded with

$$\|F\|_{B(\ell_\infty, B(\mathcal{H}))} \leq K_u.$$

Now we prove that F is ultraweakly continuous. If there is a net (a_i^λ) converges to 0 in the ultraweakly topology, then for any $(\gamma_i) \in \ell_1$, $\sum a_i^\lambda \gamma_i \rightarrow 0$. Let T belong to the trace class $S_1(\mathcal{H})$. By the polar decomposition, $T = U|T|$ where U is a partial isometry. Moreover, recall that $S_1(\mathcal{H})$ is the subset of the compact operators $K(\mathcal{H})$, $|T|$ is a self-adjoint compact operator. Thus there is an orthonormal basis (e_i) and a sequence $\lambda_i \geq 0$ so that

$$|T| = \sum \lambda_i e_i \otimes e_i$$

with $\|T\|_{S_1} = \text{tr}(|T|) = \sum_i \lambda_i < \infty$. Then for all $(a_i) \in \ell_\infty, (\gamma_j) \in \ell_1$ and $(u_j), (v_j) \subset B_1(\mathcal{H})$, we have

$$\begin{aligned}
\sum_i \sum_j |a_i \gamma_j \langle u_j, y_i \rangle \langle x_i, v_j \rangle| &\leq \| (a_i) \|_\infty \sum_j |\gamma_j| \sum_i |\langle u_j, y_i \rangle \langle x_i, v_j \rangle| \\
&= \| (a_i) \|_\infty \sum_j |\gamma_j| \sum_i \theta_{i,j} \langle u_j, y_i \rangle \langle x_i, v_j \rangle \\
&= \| (a_i) \|_\infty \sum_j |\gamma_j| \left\langle \sum_i \theta_{i,j} \langle u_j, y_i \rangle x_i, v_j \right\rangle \\
&\leq \| (a_i) \|_\infty \sum_j |\gamma_j| \sup_j \left\| \sum_i \theta_{i,j} \langle u_j, y_i \rangle x_i \right\| \\
&\leq \| (a_i) \|_\infty \sum_j |\gamma_j| K_u < \infty,
\end{aligned}$$

where $\overline{\theta_{i,j}} = \text{sgn} \{ \langle u_j, y_i \rangle \langle x_i, v_j \rangle \}$. So we have

$$\sum_i \sum_j |a_i^\lambda \lambda_j \langle Ue_j, y_i \rangle \langle x_i, e_j \rangle| < \infty$$

and

$$\sum_i \left| \sum_j \lambda_j \langle Ue_j, y_i \rangle \langle x_i, e_j \rangle \right| < \infty.$$

Therefore

$$\begin{aligned}
\text{tr} (\phi(a_i^\lambda) T) &= \sum_j \langle \phi(a_i^\lambda) T e_j, e_j \rangle \\
&= \sum_j \left\langle \phi(a_i^\lambda) U \left(\sum_k \lambda_k e_k \otimes e_k \right) e_j, e_j \right\rangle \\
&= \sum_j \langle \phi(a_i^\lambda) U \lambda_j e_j, e_j \rangle \\
&= \sum_j \lambda_j \left\langle \sum_i a_i^\lambda x_i \otimes y_i U e_j, e_j \right\rangle \\
&= \sum_j \lambda_j \sum_i a_i^\lambda \langle U e_j, y_i \rangle \langle x_i, e_j \rangle, \\
&= \sum_i a_i^\lambda \sum_j \lambda_j \langle U e_j, y_i \rangle \langle x_i, e_j \rangle,
\end{aligned}$$

which converges to 0, as claimed. \square

The main purpose of this section is to show that for every ultraweakly continuous mapping ϕ from a purely atomic abelian von Neumann algebra \mathcal{A} into $B(\mathcal{H})$, we can find a Banach space Z , an ultraweakly continuous unital homomorphism π from \mathcal{A} into $B(Z)$, and bounded linear operators T and S such that for all $a \in \mathcal{A}$,

$$\phi(a) = S\pi(a)T.$$

This result differs from Stinespring's dilation because the map ϕ here is not necessarily completely bounded and consequently the dilation space is not necessarily a Hilbert space.

While the ultraweak topology on $B(\mathcal{H})$ for a Hilbert space \mathcal{H} is well-understood, we define the ultraweak topology on $B(X)$ for a Banach space X through tensor products: Let $X \otimes Y$ be the tensor product of the Banach space X and Y . The projective norm on $X \otimes Y$ is defined by:

$$\|u\|_{\wedge} = \inf \left\{ \sum_{i=1}^n \|x_i\| \|y_i\| : u = \sum_{i=1}^n x_i \otimes y_i \right\}.$$

We will use $X \otimes_{\wedge} Y$ to denote the tensor product $X \otimes Y$ endowed with the projective norm $\|\cdot\|_{\wedge}$. Its completion will be denoted by $X \widehat{\otimes} Y$. From [R] Section 2.2, for any Banach spaces X and Y , we have the identification:

$$(X \widehat{\otimes} Y)^* = B(X, Y^*).$$

Thus $B(X, X^{**}) = (X \widehat{\otimes} X^*)^*$. Viewing $X \subseteq X^{**}$, we define the *ultraweak topology* on $B(X)$ to be the weak* topology induced by the predual $X \widehat{\otimes} X^*$. We will usually use the term *normal* to denote an ultraweakly continuous linear map.

The following lemma generalizes Theorem 4.2 and will be used in the proof of Theorem 4.4 of this section. The proof is similar to that of Theorem 4.2 and we include a sketch for completeness.

LEMMA 4.3. *Let X be a Banach space and let $E : 2^{\mathbb{N}} \rightarrow B(X)$ be an operator-valued measure on $(\mathbb{N}, 2^{\mathbb{N}})$. Denote $E(\{i\})$ by E_i for all $i \in \mathbb{N}$. Then the mapping ϕ from ℓ_{∞} into $B(X)$ defined by*

$$\phi : \ell_{\infty} \rightarrow B(X), \quad (a_i) \mapsto \sum^{SOT} a_i E_i$$

is well-defined, linear and ultraweakly continuous.

PROOF. Since $E : 2^{\mathbb{N}} \rightarrow B(X)$ is an operator-valued measure on X , we have for all $x \in X$,

$$\sum_i E_i(x)$$

converges unconditionally. Similar to the proof in Theorem 4.2, we get that

$$K_u = \sup_{x \in B_1(X)} \sup_{(\sigma_i) \subset \mathbb{D}} \left\| \sum_i \sigma_i E_i(x) \right\| < +\infty,$$

and so for all $(a_i) \in \ell_{\infty}$ and $x \in X$, we have

$$\left\| \sum_i a_i E_i(x) \right\| \leq K_u \|a_i\|_{\ell_{\infty}} \|x\|.$$

Thus ϕ is well-defined, linear and bounded with

$$\|\phi\|_{B(\ell_{\infty}, B(X))} \leq K_u.$$

For the ultraweakly continuity of ϕ , let (a_i^{λ}) be a net converging to 0 in the ultraweak topology. Then for any $(\gamma_i) \in \ell_1$, $\sum a_i^{\lambda} \gamma_i \rightarrow 0$. Let $w \in X \widehat{\otimes}_{\pi} X^*$. Then

there is a pair of sequences $(u_j, v_j) \subset X/\{0\} \times X^*/\{0\}$ with the property that $\sum \|u_j\| \|v_j\| < \infty$ and $w = \sum u_j \otimes v_j$. Thus for all $(a_i) \in \ell_\infty$, we have

$$\begin{aligned}
\sum_i \sum_j |a_i \langle E_i(u_j), v_j \rangle| &= \sum_i |a_i| \sum_j |\langle E_i(u_j), v_j \rangle| \\
&\leq \| (a_i) \|_\infty \sum_i \sum_j |\langle E_i(u_j), v_j \rangle| \\
&= \| (a_i) \|_\infty \sum_j \sum_i |\langle E_i(u_j), v_j \rangle| \\
&= \| (a_i) \|_\infty \sum_j \|u_j\| \|v_j\| \sum_i \left| \left\langle E_i \left(\frac{u_j}{\|u_j\|} \right), \frac{v_j}{\|v_j\|} \right\rangle \right| \\
&= \| (a_i) \|_\infty \sum_j \|u_j\| \|v_j\| \sum_i \theta_{i,j} \left\langle E_i \left(\frac{u_j}{\|u_j\|} \right), \frac{v_j}{\|v_j\|} \right\rangle \\
&\quad \text{here } \overline{\theta_{i,j}} = \operatorname{sgn} \left\{ \left\langle E_i \left(\frac{u_j}{\|u_j\|} \right), \frac{v_j}{\|v_j\|} \right\rangle \right\} \\
&= \| (a_i) \|_\infty \sum_j \|u_j\| \|v_j\| \left\langle \sum_i \theta_{i,j} E_i \left(\frac{u_j}{\|u_j\|} \right), \frac{v_j}{\|v_j\|} \right\rangle \\
&\leq \| (a_i) \|_\infty \sum_j \|u_j\| \|v_j\| \sup_j \left\| \sum_i \theta_{i,j} E_i \left(\frac{u_j}{\|u_j\|} \right) \right\| \\
&\leq \| (a_i) \|_\infty \sum_j \|u_j\| \|v_j\| K_u < \infty.
\end{aligned}$$

Thus we get that

$$\sum_j \sum_i |a_i^\lambda \langle E_i(u_j), v_j \rangle| = \sum_i \sum_j |a_i^\lambda \langle E_i(u_j), v_j \rangle| = \sum_i |a_i^\lambda| \sum_j |\langle E_i(u_j), v_j \rangle| < \infty$$

and

$$\sum_i \left| \sum_j \langle E_i(u_j), v_j \rangle \right| \leq \sum_i \sum_j |\langle E_i(u_j), v_j \rangle| < \infty.$$

So

$$\left(\sum_j \langle E_i(u_j), v_j \rangle \right)_i \in \ell_1.$$

Therefore

$$\begin{aligned}
\phi(a_i^\lambda)w &= \sum_j \langle \phi(a_i^\lambda)u_j, v_j \rangle \\
&= \sum_j \left\langle \sum_i a_i^\lambda E_i(u_j), v_j \right\rangle \\
&= \sum_j \sum_i a_i^\lambda \langle E_i(u_j), v_j \rangle \\
&= \sum_i a_i^\lambda \sum_j \langle E_i(u_j), v_j \rangle
\end{aligned}$$

converges to 0, as expected. \square

THEOREM 4.4. *Let H be a separable Hilbert space and $\phi : \ell_\infty(\mathbb{N}) \rightarrow B(\mathcal{H})$ such that $\phi(1) = I$ and $\phi(e_n)$ is at most rank one for all $n \in \mathbb{N}$, where $e_n = \chi_{\{n\}}$ and 1 is the function 1 in ℓ_∞ . Then the following are equivalent:*

- (i) ϕ is ultraweakly continuous,
- (ii) the induced measure E defined by $E(B) = \phi(\sum_{n \in B} e_n)$ for any $B \subseteq \mathbb{N}$ is an operator-valued measure;
- (iii) ϕ is induced by a framing (x_n, y_n) for \mathcal{H} , i.e.,

$$\phi\left(\sum_{n \in \mathbb{N}} a_n e_n\right) = \sum_{n \in \mathbb{N}}^{SOT} a_n x_n \otimes y_n$$

PROOF. (i) \Rightarrow (ii) and (i) \Rightarrow (iii) are obvious from the definition of an operator valued measure and the ultraweakly continuity of ϕ . (ii) \Rightarrow (i) follows from Lemma 4.3. (iii) \Rightarrow (i) follows from Theorem 4.2. \square

COROLLARY 4.5. Let H be a separable Hilbert space and $\phi : \ell_\infty \rightarrow B(\mathcal{H})$ such that $\phi(1) = I$ and $\phi(e_n)$ is at most rank one for all $n \in \mathbb{N}$. Then there exist a separable Banach space Z , an ultraweakly continuous unital homomorphism $\pi : \ell_\infty \rightarrow B(Z)$, and bounded linear operators $T : \mathcal{H} \rightarrow Z$ and $S : Z \rightarrow \mathcal{H}$ such that

$$\phi(a) = S\pi(a)T$$

for all $a \in \ell_\infty$, and $\pi(e_n)$ is rank one for all $n \in \mathbb{N}$.

PROOF. By Theorem 4.4, ϕ is induced by a framing (x_n, y_n) for \mathcal{H} . Thus by Theorem 4.6 in [CHL] (x_n, y_n) can be dilated an unconditional basis $\{u_n\}$ for a Banach space Z (Hence Z is separable). Let π be the induced operator valued map by (u_λ, u_λ^*) (where $\{u_\lambda^*\}$ is the dual basis of $\{u_\lambda\}$). Then π satisfies all the requirements. \square

By using our main dilation result in Chapter 2 we are able to generalize the above result to more general ultraweakly continuous operator valued mapping.

THEOREM 4.6. *Let $\phi : \ell_\infty \rightarrow B(\mathcal{H})$ be an ultraweakly continuous linear mapping. Then there exists a Banach space Z , an ultraweakly continuous unital homomorphism $\pi : \ell_\infty \rightarrow B(Z)$, and bounded linear operators $T : \mathcal{H} \rightarrow Z$ and $S : Z \rightarrow \mathcal{H}$ such that*

$$\phi(a) = S\pi(a)T$$

for all $a \in \ell_\infty$.

PROOF. Let $E : 2^{\mathbb{N}} \rightarrow B(\mathcal{H})$ be defined by

$$E(N) = \phi(\chi_N) \quad \text{for all } N \subset \mathbb{N}.$$

If (N_i) is a sequence of disjoint subsets of \mathbb{N} with union N , then it follows easily that $\sum \chi_{N_i}$ converges to χ_N under the ultraweak topology of ℓ_∞ . Since ϕ is ultraweakly continuous and $x \otimes y$ belongs to the trace class $S_1(\mathcal{H})$ for all $x, y \in \mathcal{H}$, we get that

$$\begin{aligned} \langle E(N)x, y \rangle &= \langle \phi(\chi_N)x, y \rangle = \phi(\chi_N)(x \otimes y) \\ &= \sum_i \phi(\chi_{N_i})(x \otimes y) = \sum_i \langle \phi(\chi_{N_i})x, y \rangle = \sum_i \langle E(N_i)x, y \rangle. \end{aligned}$$

Thus $E : 2^{\mathbb{N}} \rightarrow B(\mathcal{H})$ is an operator-valued measure on $(\mathbb{N}, 2^{\mathbb{N}})$. Let

$$(\Omega, \Sigma, \widetilde{M}_{E, \mathcal{M}}, \rho_{\mathcal{M}}, S_{\mathcal{M}}, T_{\mathcal{M}})$$

be its minimal dilation system. By Lemma 4.3, the mapping π from ℓ_{∞} to $B(\widetilde{M}_E)$ defined by

$$\pi : \ell_{\infty} \rightarrow B(\widetilde{M}_{E, \mathcal{M}}), \quad \pi(a_i) = \sum_i^{SOT} a_i \rho_{\mathcal{M}}(\{i\})$$

is an ultraweakly continuous unital homomorphism. Moreover, for all $(a_i) \in \ell_{\infty}$ and $x \in \mathcal{H}$, we have

$$S_{\mathcal{M}} \pi(a_i) T_{\mathcal{M}} x = S_{\mathcal{M}} \sum_i a_i \rho_{\mathcal{M}}(\{i\}) E_{x, \mathbb{N}} = S_{\mathcal{M}} \sum_i a_i E_{x, \{i\}} = \sum_i a_i E(\{i\}) x = \phi(a_i) x.$$

This completes the proof. \square

Since every separably acting purely atomic abelian von Neumann algebra is equivalent to some ℓ_{∞} via an ultraweakly continuous $*$ -isomorphism, we immediately get the following:

THEOREM 4.7. *Let \mathcal{A} be a purely atomic abelian von Neumann algebra acting on a separable Hilbert space. Then for every ultraweakly continuous linear map $\phi : \mathcal{A} \rightarrow B(\mathcal{H})$, there exists a Banach space Z , an ultraweakly continuous unital homomorphism $\pi : \ell_{\infty} \rightarrow B(Z)$, and bounded linear operators $T : \mathcal{H} \rightarrow Z$ and $S : Z \rightarrow \mathcal{H}$ such that*

$$\phi(a) = S\pi(a)T$$

for all $a \in \mathcal{A}$.

Every ultraweakly continuous linear map $\phi : L^{\infty}(\Omega, \Sigma, \mu) \rightarrow B(H)$ induces an OVM $E : (\Omega, \Sigma) \rightarrow B(H)$. Please notice that here E is absolutely continuous with respect to μ , $E \ll \mu$, that is, $\mu(E) = 0$ implies $E(E) = 0$.

LEMMA 4.8. *If $E : (\Omega, \Sigma) \rightarrow B(\mathcal{H})$ is an OVM, and if μ is a non-negative scalar measure on (Ω, Σ) such that $E \ll \mu$ (that is, E is absolutely continuous with respect to μ), then there exists a bounded linear map $\phi : L^{\infty}(\mu) \rightarrow B(\mathcal{H})$ that induces E on (Ω, Σ) , (that is, $E(B) = \phi(\chi_B)$ for all $B \in \Sigma$).*

PROOF. For any simple function $\sum_{i=1}^n \alpha_i \chi_{B_i}$ with disjoint $B_i \in \Sigma$, define $\phi(\sum_{i=1}^n \alpha_i \chi_{B_i}) = \sum_{i=1}^n \alpha_i E(B_i)$. If $\sum_{i=1}^n \alpha_i \chi_{B_i} = \sum_{j=1}^m \beta_j \chi_{A_j}$ for disjoint B_i and disjoint A_j , then $\sum_{i=1}^n \sum_{j=1}^m (\alpha_i - \beta_j) \chi_{B_i \cap A_j} = 0$. Thus, if $\alpha_i - \beta_j \neq 0$, then $\mu(B_i \cap A_j) = 0$. By $E \ll \mu$, we obtain that $\phi(\sum_{i=1}^n \alpha_i \chi_{B_i}) = \phi(\sum_{j=1}^m \beta_j \chi_{A_j})$, ϕ is well-defined on the subspace of all simple functions in $L^{\infty}(\mu)$. To prove that ϕ is linear, we only need to notice that $\sum_{i=1}^n \alpha_i \chi_{B_i} + \sum_{j=1}^m \beta_j \chi_{A_j} = \sum_{i=1}^n \sum_{j=1}^m (\alpha_i + \beta_j) \chi_{B_i \cap A_j}$. For the boundedness, we need the following uniformly boundedness first, since $\sup_{E \in \Sigma} \|E(B)\| < \infty$, it is easy to obtain that

$$\sup_{\{B_i\}_{i=1}^n \text{ is a partition of } \Omega} \sup_{\|x\|, \|y\| \leq 1} \sum_{i=1}^n |\langle E(B_i)x, y \rangle| = M_{\phi} < \infty.$$

Then we have

$$\begin{aligned}
\left\| \phi \left(\sum_{i=1}^n \alpha_i \chi_{B_i} \right) \right\| &= \left\| \sum_{i=1}^n \alpha_i E(B_i) \right\| \\
&= \sup_{\|x\|, \|y\| \leq 1} \sum_{i=1}^n |\alpha_i \langle E(B_i)x, y \rangle| \\
&\leq \left(\sup_{\|x\|, \|y\| \leq 1} \sum_{i=1}^n |\langle E(B_i)x, y \rangle| \right) \max_{1 \leq i \leq n} |\alpha_i| \\
&\leq M_\phi \left\| \sum_{i=1}^n \alpha_i \chi_{B_i} \right\|_{L^\infty}.
\end{aligned}$$

Because the simple functions are dense in $L^\infty(\mu)$, the conclusion follows. \square

For an OVM $E : (\Omega, \Sigma) \rightarrow B(H)$, where H is a separable Hilbert space, there always exists a non-negative scalar measure μ on (Ω, Σ) such that $E \ll \mu$ holds. Then we immediately get that

COROLLARY 4.9. If $E : (\Omega, \Sigma) \rightarrow B(\mathcal{H})$ is an OVM, then there exists μ which is a non-negative scalar measure on (Ω, Σ) such that $E \ll \mu$, and a bounded linear map $\phi : L^\infty(\mu) \rightarrow B(\mathcal{H})$ that induces E on (Ω, Σ) .

We remark that the conclusion of Corollary 4.9 actually holds for more general von Neumann algebras [BW]: Let \mathcal{A} be a von Neumann algebra without direct summand of type I_2 . Then every bounded and finitely additive $B(H)$ -valued measure (c.f. [BW] for definition) on the projection lattice of \mathcal{A} can be uniquely extend to bounded linear map from \mathcal{A} to $B(H)$. We include Lemma 4.8 and Corollary 4.9 here since they are directly related to the following problem. We think if the answer to Problem C is positive then a solution might be along the lines of the proof of Lemma 4.8.

Problem C. Let $E : (\Omega, \Sigma) \rightarrow B(\mathcal{H})$ be an OVM. Is there an ultraweakly continuous map $\phi : L^\infty(\mu) \rightarrow B(\mathcal{H})$ that induces E on (Ω, Σ) ?

4.3. The Noncommutative Case

THEOREM 4.10. [Banach Algebra Dilation Theorem] Let \mathcal{A} be a Banach algebra, and let $\phi : \mathcal{A} \rightarrow B(\mathcal{H})$ be a bounded linear operator, where \mathcal{H} is a Banach space. Then there exists a Banach space Z , a bounded linear unital homomorphism $\pi : \mathcal{A} \rightarrow B(Z)$, and bounded linear operators $T : \mathcal{H} \rightarrow Z$ and $S : Z \rightarrow \mathcal{H}$ such that

$$\phi(a) = S\pi(a)T$$

for all $a \in \mathcal{A}$.

PROOF. Consider the algebraic tensor product space $\mathcal{A} \otimes \mathcal{H}$, and define a $B(\mathcal{A}, \mathcal{H})$ operator tensor norm with respect to ϕ as follows: for any $a \in \mathcal{A}$ and $x \in \mathcal{H}$, identify $a \otimes x$ with the map $a \rightarrow \mathcal{H}$ defined by

$$(a \otimes x)(b) = \phi(ba)x.$$

Then $a \otimes x$ is a bounded linear operator from \mathcal{A} to \mathcal{H} with $\|a \otimes x\|_{B(\mathcal{A}, \mathcal{H})} \leq \|\phi\| \|a\| \|x\|$. So this defines a quasi-norm on $\mathcal{A} \otimes \mathcal{H}$. Let

$$\mathcal{N} = \left\{ \sum_i c_i a_i \otimes x_i : \left\| \sum_i c_i a_i \otimes x_i \right\|_{B(\mathcal{A}, \mathcal{H})} = 0 \right\}$$

and $\mathcal{A} \widetilde{\otimes}_\phi \mathcal{H} / \mathcal{N}$ be the completion of the norm space $\mathcal{A} \otimes \mathcal{H} / \mathcal{N}$.

For any $a \in \mathcal{A}$, let us define $\pi(a) : \mathcal{A} \otimes \mathcal{H} / \mathcal{N} \rightarrow \mathcal{A} \otimes \mathcal{H} / \mathcal{N}$ by

$$\pi(a) \left(\sum_i a_i \otimes x_i \right) = \sum_i (a a_i) \otimes x_i.$$

Assume that $f = \sum_{i=1}^n a_i \otimes x_i = \sum_{j=1}^m b_j \otimes y_j$. Then we have

$$\begin{aligned} \pi(a) \left(\sum_i a_i \otimes x_i \right) (b) &= \left(\sum_i (a a_i) \otimes x_i \right) (b) \\ &= \sum_i \phi(b a a_i) x_i \\ &= \left(\sum_i a_i \otimes x_i \right) (b a) \\ &= f(b a), \end{aligned}$$

and

$$\begin{aligned} \pi(a) \left(\sum_j b_j \otimes y_j \right) (b) &= \left(\sum_j (a b_j) \otimes y_j \right) (b) \\ &= \sum_j \phi(b a b_j) y_j \\ &= \left(\sum_j b_j \otimes y_j \right) (b a) \\ &= f(b a). \end{aligned}$$

Therefor $\pi(a)$ is well defined.

The boundedness $\pi(a)$ (with $\|\pi(a)\| \leq \|a\|$) follows from

$$\begin{aligned}
\|\pi(a)(f)\| &= \left\| \pi(a) \left(\sum_i a_i \otimes x_i \right) \right\| \\
&= \left\| \sum_i (aa_i) \otimes x_i \right\| \\
&= \sup_{b \in B_{\mathcal{A}}} \left\| \left(\sum_i (aa_i) \otimes x_i \right) (b) \right\| \\
&= \sup_{b \in B_{\mathcal{A}}} \left\| \sum_i \phi(baa_i)x_i \right\| \\
&= \|a\| \sup_{b \in B_{\mathcal{A}}} \left\| \sum_i \phi(b \frac{a}{\|a\|} a_i)x_i \right\| \\
&\leq \|a\| \sup_{b \in B_{\mathcal{A}}} \left\| \sum_i \phi(ba_i)x_i \right\| \\
&= \|a\| \sup_{b \in B_{\mathcal{A}}} \left\| \left(\sum_i a_i \otimes x_i \right) (b) \right\| \\
&= \|a\| \left\| \sum_i a_i \otimes x_i \right\| \\
&= \|a\| \cdot \|f\|.
\end{aligned}$$

Extend $\pi(a)$ to a bounded linear operator on $\mathcal{A} \widetilde{\otimes}_{\phi} \mathcal{H}/\mathcal{N}$ which we still denote it by $\pi(a)$. Thus $\pi : \mathcal{A} \rightarrow \mathcal{A} \widetilde{\otimes}_{\phi} \mathcal{H}/\mathcal{N}$ is a bounded linear operator. Moreover,

$$\begin{aligned}
\pi(ab) \left(\sum_i a_i \otimes x_i \right) &= \sum_i (aba_i) \otimes x_i \\
&= \pi(a) \left(\sum_i (ba_i) \otimes x_i \right) \\
&= \pi(a)\pi(b) \left(\sum_i a_i \otimes x_i \right).
\end{aligned}$$

Hence

$$\pi(ab) = \pi(a)\pi(b),$$

and therefore π is a homomorphism.

Define $T : \mathcal{H} \rightarrow \mathcal{A} \widetilde{\otimes}_{\phi} \mathcal{H}/\mathcal{N}$ by $T(x) = \mathbf{1} \otimes x$. Since

$$\begin{aligned}
\|T(x)\| &= \|\mathbf{1} \otimes x\| \\
&= \sup_{a \in B_{\mathcal{A}}} \|(\mathbf{1} \otimes x)(a)\| \\
&= \sup_{a \in B_{\mathcal{A}}} \|\phi(a)x\| \\
&\leq \sup_{a \in B_{\mathcal{A}}} \|a\| \|\phi\| \|x\| \\
&= \|\phi\| \|x\|,
\end{aligned}$$

we have that T is a bounded linear operator with $\|T\| \leq \|\phi\|$.

Define $S : \mathcal{A} \otimes \mathcal{H}/\mathcal{N} \rightarrow \mathcal{H}$ by $S(a \otimes x) = E(a)x$ and linearly extend S to $\mathcal{A} \otimes \mathcal{H}/\mathcal{N}$. It is easy to check that S is well-defined. Since

$$\begin{aligned} \|S(a \otimes x)\| &= \|\phi(a)x\| \\ &\leq \sup_{b \in B_{\mathcal{A}}} \|\phi(ba)x\| \\ &= \sup_{b \in B_{\mathcal{A}}} \|(a \otimes x)(b)\| \\ &= \|a \otimes x\|, \end{aligned}$$

we have that S is a bounded linear operator with $\|S\| \leq 1$. Extend S to a bounded linear operator from $\widetilde{\mathcal{A} \otimes_{\phi} \mathcal{H}}/\mathcal{N}$ to \mathcal{H} , which we still denote it by S .

Finally, for any $x \in \mathcal{H}$ we have

$$\begin{aligned} S\pi(a)T(x) &= S\rho(a)(\mathbf{1} \otimes x) \\ &= S(a \otimes x) \\ &= \phi(a)x. \end{aligned}$$

Thus $\phi(a) = S\pi(a)T$. □

REMARK 4.11. Consider the construction of the dilation space in Theorem 4.10, and compare it with the construction of the dilation space in Theorem 4.7. In the first case we take the algebraic tensor product $\mathcal{A} \otimes \mathcal{H}$ considered as a linear space of operators in $B(\mathcal{A}, \mathcal{H})$ to obtain a quasi-norm on $\mathcal{A} \otimes \mathcal{H}$ which we denote $\|\cdot\|_{B(\mathcal{A}, \mathcal{H})}$. Then we mod out by the kernel to obtain a norm on the quotient space, and then we complete it to obtain a Banach space. If we began this construction by using a dense subalgebra of \mathcal{A} instead of \mathcal{A} itself, then the construction goes through smoothly, and the dilation space is the same. In the case of the second space we work with $\mathcal{A} = \ell_{\infty}$, and the reader can note that we start with the algebraic tensor product of the dense subalgebra $\mathcal{A}_0 = \{\sum c_i \chi_{E_i} : c_i \in \mathbb{C}, E_i \in \Sigma\}$ with \mathcal{H} , and define a special “minimal” quasi-norm $\|\cdot\|_1$ on it as in Section 2.4, then mod out by the kernel, and complete it to obtain the dilation space. In both cases we could begin with the dense subalgebra \mathcal{A}_0 of ℓ_{∞} , form the algebraic tensor product $\mathcal{A}_0 \otimes \ell_{\infty}$, and put a quasi-norm on the space. It is easy to show that these two quasi-norms are equivalent in the sense that each one is dominated by a constant multiple of the other. Hence the dilation spaces of Theorem 4.7 and Theorem 4.10 are equivalent for the special case $\mathcal{A} = \ell_{\infty}$.

As with Stinespring’s dilation theorem, if A and \mathcal{H} in Theorem 4.10 are both separable then the dilated Banach space Z is also separable. However, the Banach algebras we are interested in include von Neumann algebras and these are generally not separable, and the linear maps $\phi : A \rightarrow B(H)$ are often normal. So we pose the following two problems.

Problem D. Let K, H be separable Hilbert spaces, let $A \subset B(K)$ be a von Neumann algebra, and let $\phi : A \rightarrow B(H)$ be a bounded linear map. When is there a *separable* Banach space Z , a bounded linear unital homomorphism $\pi : \mathcal{A} \rightarrow B(Z)$, and bounded linear operators $T : \mathcal{H} \rightarrow Z$ and $S : Z \rightarrow \mathcal{H}$ such that

$$\phi(a) = S\pi(a)T$$

for all $a \in \mathcal{A}$?

Problem E. Let $A \subset B(K)$ be a von Neumann algebra, and $\phi : A \rightarrow B(H)$ be a normal linear map. When can we dilate ϕ to a *normal* linear unital homomorphism $\pi : \mathcal{A} \rightarrow B(Z)$ for some Banach space Z ?

Although we do not know the answer to Problem E, we do have the following result:

THEOREM 4.12. *Let K, H be Hilbert spaces, $A \subset B(K)$ be a von Neumann algebra, and $\phi : A \rightarrow B(H)$ be a bounded linear operator which is ultraweakly-SOT continuous on the unit ball B_A of A . Then there exists a Banach space Z , a bounded linear homeomorphism $\pi : A \rightarrow B(Z)$ which is SOT-SOT continuous on B_A , and bounded linear operator $T : H \rightarrow Z$ and $S : Z \rightarrow H$ such that*

$$\phi(a) = S\pi(a)T$$

for all $a \in A$.

PROOF. We only need to prove that the bounded linear homeomorphism π constructed in Theorem 4.10 is SOT-SOT continuous.

Assume that a net $\{a_\lambda\} \subset B_A$. It is sufficient to prove that $a_\lambda \rightarrow 0$ (SOT) implies that $\pi(a_\lambda) \rightarrow 0$ (SOT). Since $\{\pi(a_\lambda)\}$ is norm-bounded, we have that $\pi(a_\lambda) \rightarrow 0$ (SOT) if and only if $\pi(a_\lambda)\xi \rightarrow 0$ in norm for a dense set of ξ . Thus, we only need to prove that

$$\pi(a_\lambda)(a \otimes x) = (a_\lambda a) \otimes x \rightarrow 0$$

in the norm topology for all $a \in A$ and $x \in H$. If this is not the case, then there exist $\delta > 0$, a subnet of $\{a_\lambda\}$ (still denoted by $\{a_\lambda\}$), and $\{b_\lambda\} \subset B_A$ such that

$$\|(a_\lambda a \otimes x)(b_\lambda)\| = \|\varphi(b_\lambda a_\lambda a)(x)\| > \delta.$$

Since on any norm-bounded set the WOT and ultraweak topology are the same (and in particular the unit ball is compact in both topologies), there is a subnet of $\{b_\lambda\}$ (still denoted by $\{b_\lambda\}$) converges to some element $b \in B_A$ in the ultraweak topology (or WOT). Thus, from our hypothesis that $a_\lambda \rightarrow 0$ (SOT), we get that $b_\lambda a_\lambda a \rightarrow 0$ (SOT). This implies that $b_\lambda a_\lambda a \rightarrow 0$ in ultraweak topology (or WOT), and therefore $\varphi(b_\lambda a_\lambda a) \rightarrow 0$ (SOT) since ultraweakly-SOT continuous on B_A . This leads to a contradiction. \square

COROLLARY 4.13. Let K, H be separable Hilbert spaces, $A \subset B(K)$ be a von Neumann algebra, and $\phi : A \rightarrow B(H)$ be a bounded linear operator which is ultraweakly-SOT continuous on B_A . Then there exists a separable Banach space \tilde{Z} , a bounded linear homeomorphism $\tilde{\pi} : A \rightarrow B(\tilde{Z})$ which is SOT-SOT continuous on B_A , and bounded linear operator $\tilde{T} : H \rightarrow \tilde{Z}$ and $\tilde{S} : \tilde{Z} \rightarrow H$ such that

$$\phi(a) = \tilde{S}\tilde{\pi}(a)\tilde{T}$$

for all $a \in A$.

PROOF. By Theorem 4.12, there exists a Banach space Z , a bounded linear homeomorphism $\pi : A \rightarrow B(Z)$ which is SOT-SOT continuous on B_A , and bounded linear operator $T : H \rightarrow Z$ and $S : Z \rightarrow H$ such that $\phi(a) = S\pi(a)T$ for all $a \in A$. Since K is separable, there is $\{a_i\}_{i=1}^\infty$ SOT dense in B_A . Define

$$\tilde{Z} = \overline{\text{span}}^{\|\cdot\|} \{\pi(a_i)T(h) : 1 \leq i < \infty, h \in H\}.$$

Then \tilde{Z} is separable since H is separable. Moreover, the SOT-SOT continuity of π (restricted to the unit ball of A) implies that $\pi(A)\tilde{Z} \subset \tilde{Z}$ and $\pi(A)T(H) \subset \tilde{Z}$. Let $V : \tilde{Z} \rightarrow Z$ be the inclusion linear map. Then

$$\phi(a) = SV^{-1}\pi(a)VT, \quad \forall a \in A.$$

Let $\tilde{\pi}(a) = V^{-1}\pi(a)V$ acting on \tilde{Z} . Then $\tilde{\pi}$ is a dilation of ϕ on the separable Banach space \tilde{Z} and $\tilde{\pi} : A \rightarrow B(\tilde{Z})$ is also SOT-SOT continuous when restricted to the unit ball of A . \square

REMARK 4.14. Dilations and the similarity problem: Let \mathcal{A} be a C^* -algebra. In 1955, Kadison [K1] formulated the following still open conjecture: Any bounded homomorphism π from a C^* -algebra \mathcal{A} into the algebra $B(H)$ of all bounded operators on a Hilbert space H is similar to a $*$ -homomorphism, i.e. there is an invertible operator $T \in B(H)$ such that $T\pi(\cdot)T^{-1}$ is a $*$ -homomorphism from \mathcal{A} to $B(H)$. This problem is known to be equivalent to several famous open problems (c.f. [Pi]) including the derivation problem: Is every derivation from a C^* -algebra $\mathcal{A} \subseteq B(H)$ into $B(H)$ inner? While Kadison's similarity problem remains unsettled, many remarkable partial results are known. In particular, it is well known that π is similar to a $*$ -homomorphism if and only if it is completely bounded. It has been proved that a bounded unital homomorphism $\pi : \mathcal{A} \rightarrow B(H)$ is completely bounded (and hence similar to a $*$ -representation) if \mathcal{A} is nuclear [Bun, Chr]; or if $\mathcal{A} = B(H)$, or more generally if \mathcal{A} has no tracial states; or if \mathcal{A} is commutative; or if \mathcal{A} is a II_1 -factor with Murray and von Neumann's property Γ ; or if π is cyclic [Haa]. Therefore if \mathcal{A} belongs to any of the above mentioned classes, and $\phi : \mathcal{A} \rightarrow B(H)$ is a bounded but not completely bounded linear map, then the dilation space Z in Theorem 4.10 [Banach algebra dilation theorem] can never be a Hilbert space since otherwise $\pi : \mathcal{A} \rightarrow B(Z)$ would be completely bounded and so would be ϕ . On the other hand, if there is a non completely bounded map ϕ from a C^* -algebra to $B(H)$ that has a Hilbert space dilation: $\pi : \mathcal{A} \rightarrow B(Z)$ (i.e., where Z is a Hilbert space), then it would be a counterexample to the Kadison's similarity problem. So we have the following question: Is there a non-cb map that admits a Hilbert space dilation to a bounded homomorphism?

REMARK 4.15. For a countable index set Λ , there is a 1-1 correspondence between the set of (discrete) framings on a Hilbert space \mathcal{H} indexed by Λ and the set of ultraweakly continuous unital linear maps from $\ell_\infty(\Lambda)$ into $B(\mathcal{H})$. Here, unital means it takes the function 1 in $\ell_\infty(\Lambda)$ to the identity operator in $B(\mathcal{H})$. There is also a 1-1 correspondence between the set of (discrete) framings on a Hilbert space \mathcal{H} indexed by Λ and the set of purely atomic probability operator-valued measures on the σ -algebra of all subsets of Λ with rank-1 atoms in $B(\mathcal{H})$. So we have a space of ordered triples which consists of a discrete framing, a purely atomic operator-valued probability measure with rank-1 atoms, and an ultraweakly continuous unital linear map from an purely atomic abelian von Neumann algebra into $B(\mathcal{H})$ where each minimal projection is sent to a rank ≤ 1 operator. Each item in a triple determines the other items uniquely. There is a consistent dilation theory, where the dilation of the discrete framing, the operator-valued measure, and the ultraweakly continuous unital linear map, all have the same dilation space and the dilation procedure commutes with the correspondences in the triple, i.e. when we go from framing to OVM to linear map and then dilate each one to get a dilated

triple, it will be the same as if we dilate any one and then derived the other two from it in the natural way. Similarly for a more general index set Λ (i.e. compact or locally compact, whatever is more suitable) there are connections among the set of operator-valued probability measures on Λ taking values in $B(\mathcal{H})$ for a Hilbert space \mathcal{H} , the set of unital ultraweakly continuous maps from $\ell_\infty(\Lambda)$ into $B(\mathcal{H})$, and their corresponding dilation theory. Although not every ultraweakly continuous unital linear mapping is associated with a (continuous) framing, there are many ultraweakly continuous unital linear mappings that are induced by framings. For this reason we can view ultraweakly continuous unital linear mappings from $\ell_\infty(\Lambda)$ to $B(\mathcal{H})$ as *abstract framings*, which is the commutative theory because the domain of the map is a commutative von neumann algebra. When we pass to the noncommutative domains, the dilation theory for various linear maps can be viewed as a noncommutative (abstract) framing dilation theory.

CHAPTER 5

Examples

We provide the details of the two examples mentioned in the introduction chapter. Our first example shows that there exists a framing for a Hilbert space whose induced operator valued measure fails to admit a Hilbert space dilation. Equivalently, it cannot be re-scaled to obtain a framing that admits a Hilbert space dilation. The construction is based on an example of Ozaka [Os] of a normal non-completely bounded map of $\ell^\infty(\mathbb{N})$ into $B(H)$.

THEOREM 5.1. *There exist a framing for a Hilbert space such that its induced operator-valued measure is not completely bounded, and consequently it can not be re-scaled to obtain a framing that admits a Hilbert space dilation.*

Since if a framing admits a Hilbert space dilation, then the induced operator valued map is completely bounded and a re-scaled framing induces the same operator valued map, we only need to show that there exists a framing for a Hilbert space such that its induced operator-valued measure is not completely bounded. We need the following lemmas.

LEMMA 5.2. *Let $\{A_n\}$ be a sequence of finite-rank bounded linear operators on a Hilbert space H such that*

- (i) $A_n A_m = A_m A_n = 0$ for all $n \neq m$;
- (ii) *there exist mutually orthogonal projections $\{P_n\}$ such that $A_n = P_n A_n P_n$ for all n .*
- (iii) $\sum_{n=1}^{\infty} A_n$ *converges unconditionally to $A \in B(H)$ with the strong operator topology.*

Assume that A_n has the rank one operator decomposition $A_n = Q_{n,1} + \dots + Q_{n,k_n}$ such that for any subset Λ of $J_n := \{(n,1), \dots, (n,k_n)\}$ we have $\|\sum_{j \in \Lambda} Q_j\| \leq \|A_n\|$. Let J be the disjoint union of J_n ($n = 1, 2, \dots$). Then the series $\sum_{j \in J} Q_j$ converges unconditionally to A .

PROOF. Let $x \in H$. Without losing the generality we can assume that $x \in PH$ where $P = \sum_{n=1}^{\infty} P_n$. Let $\epsilon > 0$. Then there exists N such that

$$\left\| \sum_{n=1}^N P_n x - x \right\| < \epsilon/2 \|A\|.$$

Now let $\{j_\ell\}_{\ell=1}^{\infty}$ be an enumeration of J and let L be such that $\{j_1, \dots, j_L\}$ contain $\cup_{n=1}^N J_n$. Write $x = x_1 + x_0$ with $x_1 = \sum_{n=1}^N P_n x$ and $x_0 = (\sum_{n=1}^N P_n)^\perp x$. Then for any $L' \geq L$, we get

$$\sum_{\ell=1}^{L'} Q_{j_\ell} x = \sum_{n=1}^N \sum_{j_\ell \in J_n} Q_{j_\ell} x + \sum_{j_\ell \notin J_n, \ell \leq L'} Q_{j_\ell} x = \sum_{n=1}^N A_n x + \sum_{j_\ell \notin J_n, \ell \leq L'} Q_{j_\ell} x_0$$

where we use the property that $\sum_{j_\ell \notin J_n, \ell \leq L'} Q_{j_\ell} x_1 = 0$. Moreover, from our assumptions on $\{A_n\}$ and their rank-one decompositions we also have

$$\left\| \sum_{j_\ell \notin J_n, \ell \leq L'} Q_{j_\ell} x_0 \right\| \leq \sup\{\|A_n\|\} \leq \|A\|$$

and

$$\sum_{n=1}^N A_n x = A \sum_{n=1}^N P_n x.$$

Therefor we get

$$\begin{aligned} \left\| \sum_{\ell=1}^{L'} Q_{j_\ell} x - Ax \right\| &\leq \left\| A \sum_{n=1}^N P_n x - Ax \right\| + \left\| \sum_{j_\ell \notin J_n, \ell \leq L'} Q_{j_\ell} x_0 \right\| \\ &\leq \|A\| \cdot \left\| \sum_{n=1}^N P_n x - x \right\| + \|A\| \cdot \|x_0\| \\ &= 2\|A\| \cdot \|x_0\| < \epsilon. \end{aligned}$$

This completes the proof. \square

LEMMA 5.3. *Let A be a rank- k operator. Then there exists rank-one decomposition $A = \sum_{i=1}^k Q_i$ such that $\|\sum_{i \in I} Q_i\| \leq \|A\|$ for any subset I of $\{1, 2, \dots, k\}$.*

PROOF. Let $A = U|A|$ be its polar decomposition and write $|A| = \sum_{i=1}^k x_i \otimes x_i$. Then we have $\|\sum_{i \in I} x_i \otimes x_i\| \leq \|\sum_{i=1}^k x_i \otimes x_i\| = \|A\|$. Let $Q_i = U x_i \otimes x_i$. Then we get

$$\begin{aligned} \left\| \sum_{i \in I} Q_i \right\| &= \left\| U \sum_{i \in I} x_i \otimes x_i \right\| \\ &\leq \|U\| \cdot \left\| \sum_{i \in I} x_i \otimes x_i \right\| \\ &\leq \left\| \sum_{i \in I} x_i \otimes x_i \right\| \\ &\leq \left\| \sum_{i=1}^k x_i \otimes x_i \right\| = \|A\| \end{aligned}$$

holds for any subset I of $\{1, 2, \dots, k\}$, as claimed. \square

Now we prove Theorem 5.1: By Lemma 2.4 (or Lemma 2.1) in [Os] there exists a σ -weakly continuous bounded linear map ψ from $\oplus_{n=1}^\infty \ell_n^\infty$ into $\oplus_{n=1}^\infty M_{2^n}$ (as a subalgebra acting on $H := \oplus \mathbb{C}^{2^n}$) which is not completely bounded and $\psi = \oplus_{n=1}^\infty \psi_n$, where $\psi_n : \ell_n^\infty \rightarrow M_{2^n}$ and M_{2^n} is the algebra of $2^n \times 2^n$ -matrices. Let $B_n = \sum_{i=1}^n \psi(e_i^n)$, where $e_i^n \in \ell_n^\infty$ such that $e_i^n(j) = \delta_{i,j}$, and let $B = \psi(e)$ where $e \in \oplus_{n=1}^\infty \ell_n^\infty$ is the unital element. Let P_n be the orthogonal projections from H onto \mathbb{C}^{2^n} . Without losing the generality we can assume that $\|\psi\| < 1$. Let $A_n = B_n + P_n$. Then $\{A_n\}$ satisfies all the conditions (i)–(iii) in Lemma 5.2, and $\sum_{n=1}^\infty A_n$ converges to $A = B + I$ unconditionally in the strong operator topology. By Lemma 5.3, we can decompose each $A_n = Q_{n,1} + \dots + Q_{n,k_n}$ such that for any subset Λ of $J_n := \{(n, 1), \dots, (n, k_n)\}$ we have $\|\sum_{j \in \Lambda} Q_j\| \leq \|A_n\|$. Thus, by Lemma 5.2, $\sum_{j \in J} Q_j$ converges unconditionally to A . Write $Q_j = x_j \otimes y_j$, and

let $u_j = A^{-1}x_j$. Then $\{u_j, y_j\}$ form a framing for H since $\sum_{j \in J} u_j \otimes y_j$ converges unconditionally to I in the strong operator topology.

We claim that the induced operator-valued map $\Phi : \ell^\infty(J) \rightarrow B(H)$ is not completely bounded. In fact, if it is completely bounded, then operator valued map Ψ by $\{x_j, y_j\}$ will be completely bounded. By embedding $\oplus_{n=1}^\infty \ell_n^\infty$ naturally to $\ell^\infty(J)$, we can view $\oplus_{n=1}^\infty \ell_n^\infty$ as a subalgebra of $\ell^\infty(J)$, and so the restriction of Ψ to this subalgebra is $\psi + id$, where id is the natural embedding of $\oplus_{n=1}^\infty \ell_n^\infty$ into $B(H)$. This $\psi + id$ is completely bounded and so ψ is completely bounded since the embedding map id is completely bounded. This leads to a contradiction. Therefore $\Phi : \ell^\infty(J) \rightarrow B(H)$ is not completely bounded, and thus $\{u_j, y_j\}$ it can not be re-scaled to obtain a framing that admits a Hilbert space dilation. \square

Next we examine Example 3.9 of [CHL], which gave the first apparently non-trivial example of a bounded framing for a Hilbert space which is not a dual pair of frames. A natural, geometric, Banach space dilation of it was constructed in [CHL], along with a proof that it did not have any Hilbert dilation space. Both the construction and the proof was nontrivial, and both were sketched out in [CHL] without providing great detail. It was thought for a long time that this example might be a key to this subject. But when we were finalizing this paper we found a proof that this example can, in fact, be *re-scaled* to give a dual pair of frames, in fact a pair which consists of two copies of a single Parseval frame. Since this example was a guiding example for a lengthy time, we include our analysis of it. After obtaining this result, since we knew we still needed a true limiting example for this theory, we managed to obtain the example in Theorem 5.1 above. The [CHL] example points out how extremely difficult it can be in general to determine when a given framing is scalable to a dual frame pair.

THEOREM 5.4. *Example 3.9 in [CHL] can be re-scaled to a dual pair of frames.*

PROOF. This result is very technical and it is a stand-alone result for the present paper because the details of the proof are not used anywhere else in this paper. So we must refer the reader to [CHL] for some of the terminology and Banach space background. With this in hand this proof can be handily worked through.

Fix any $1 < p < \infty$ and $p \neq 2$ and natural number n . Let $\{e_i^n\}_{i=1}^{2^n}$ be a unconditional unit basis of $\ell_p^{2^n}$ and $\{(e_i^n)^*\}_{i=1}^{2^n}$ be the dual of $\{e_i^n\}_{i=1}^{2^n}$. We denote the Rademacher vectors in $\ell_p^{2^n}$ by $\{r_i^n\}_{i=1}^n$, where

$$r_i^n = \frac{1}{2^{n/p}} \sum_{j=1}^{2^n} \epsilon_{ij} e_j^n$$

with $\|r_i^n\|_{\ell_p^{2^n}} = 1$ and ϵ_{ij} satisfying the condition

$$\sum_{j=1}^{2^n} \epsilon_{ij} \epsilon_{kj} = \delta_{ik} \cdot 2^n.$$

Then $\{r_i^n\}_{i=1}^n$ are linearly independent because they are the Rademacher vectors. So $W_n = \text{span}\{r_i^n\}_{i=1}^n$ is a n -dimensional subspace of $\ell_p^{2^n}$. For any $\sum_{i=1}^n a_i r_i^n \in W_n$,

we have

$$\begin{aligned}
 \left\| \sum_{i=1}^n a_i r_i^n \right\|_{\ell_p^{2^n}} &= \left\| \sum_{i=1}^n \frac{a_i}{2^{n/p}} \sum_{j=1}^{2^n} \epsilon_{ij} e_j^n \right\|_{\ell_p^{2^n}} \\
 &= \frac{1}{2^{n/p}} \left\| \sum_{j=1}^{2^n} \left(\sum_{i=1}^n a_i \epsilon_{ij} \right) e_j^n \right\|_{\ell_p^{2^n}} \\
 (5.1) \quad &= \frac{1}{2^{n/p}} \left(\sum_{j=1}^{2^n} \left| \sum_{i=1}^n a_i \epsilon_{ij} \right|^p \right)^{1/p}.
 \end{aligned}$$

Let $\text{sign}(\sin 2^i \pi t)$, $i = 0, 1, \dots, n$ be the Rademacher functions on $[0, 1]$. By Theorem 2.b.3 in [LT], we have

$$\begin{aligned}
 \int_0^1 \left| \sum_{i=1}^n a_i \text{sign}(\sin 2^i \pi t) \right|^p dt &= \sum_{j=1}^{2^n} \int_{\frac{j-1}{2^n}}^{\frac{j}{2^n}} \left| \sum_{i=1}^n a_i \text{sign}(\sin 2^i \pi t) \right|^p dt \\
 &= \sum_{j=1}^{2^n} \int_{\frac{j-1}{2^n}}^{\frac{j}{2^n}} \left| \sum_{i=1}^n a_i \epsilon_{ij} \right|^p dt \\
 &= \sum_{j=1}^{2^n} \frac{1}{2^n} \left| \sum_{i=1}^n a_i \epsilon_{ij} \right|^p.
 \end{aligned}$$

Let A_p, B_p be the constants as in Theorem 2.b.3 in [LT]. Then for any $\{a_i\}_{i=1}^n$, we have

$$A_p \left(\sum_{i=1}^n |a_i|^2 \right)^{1/2} \leq \left(\sum_{j=1}^{2^n} \frac{1}{2^n} \left| \sum_{i=1}^n a_i \epsilon_{ij} \right|^p \right)^{1/p} \leq B_p \left(\sum_{i=1}^n |a_i|^2 \right)^{1/2},$$

that is

$$A_p \left(\sum_{i=1}^n |a_i|^2 \right)^{1/2} \leq \frac{1}{2^{n/p}} \left(\sum_{j=1}^{2^n} \left| \sum_{i=1}^n a_i \epsilon_{ij} \right|^p \right)^{1/p} \leq B_p \left(\sum_{i=1}^n |a_i|^2 \right)^{1/2}.$$

Thus from equation (5.1), we obtain

$$A_p \left(\sum_{i=1}^n |a_i|^2 \right)^{1/2} \leq \left\| \sum_{i=1}^n a_i r_i^n \right\|_{\ell_p^{2^n}} \leq B_p \left(\sum_{i=1}^n |a_i|^2 \right)^{1/2}.$$

Let \mathcal{H}_n be the span of $\{r_1^n, \dots, r_n^n\}$ with the inner product

$$\left\langle \sum_{i=1}^n a_i r_i^n, \sum_{k=1}^n b_k r_k^n \right\rangle = \sum_{i=1}^n a_i \bar{b}_i.$$

The induced Hilbert space norm is

$$\left\| \sum_{i=1}^n a_i r_i^n \right\|_{\mathcal{H}_n} = \left(\sum_{i=1}^n |a_i|^2 \right)^{1/2},$$

and $\{r_1^n, \dots, r_n^n\}$ is an orthonormal basis for \mathcal{H}_n . Then \mathcal{H}_n is a Hilbert space isometrically isomorphism to l_2^n . The spaces \mathcal{H}_n and W_n are naturally isomorphism as Banach space, being the same vector space with two different norms. Let $U_n : W_n \rightarrow \mathcal{H}_n$, $U_n x = x$ be this isomorphism.

Define a mapping $P_n : \ell_p^{2^n} \rightarrow W_n$ by the dual Rademachers

$$(r_i^n)^* = \frac{1}{2^{n/q}} \sum_{j=1}^{2^n} \epsilon_{ij} (e_j^n)^*,$$

where $1/p + 1/q = 1$. That is $P_n(x) = \sum_{i=1}^n (r_i^n)^*(x) r_i^n$ for any $x \in \ell_p^{2^n}$. So we have $P_n(e_k^n) = \sum_{i=1}^n (r_i^n)^*(e_k^n) r_i^n$. Since

$$(r_i^n)^*(e_k^n) = \frac{1}{2^{n/q}} \sum_{j=1}^{2^n} \epsilon_{ij} (e_j^n)^*(e_k^n) = \frac{1}{2^{n/q}} \epsilon_{ik},$$

we get that

$$(5.2) \quad P_n(e_k^n) = \sum_{i=1}^n \frac{1}{2^{n/q}} \epsilon_{ik} r_i^n = \frac{1}{2^{n/q}} \sum_{i=1}^n \epsilon_{ik} r_i^n$$

and

$$P_n^2(e_k^n) = P_n \left(\frac{1}{2^{n/q}} \sum_{i=1}^n \epsilon_{ik} r_i^n \right) = \frac{1}{2^{n/q}} \sum_{i=1}^n \epsilon_{ik} P_n(r_i^n).$$

On the other hand, since $\sum_{j=1}^{2^n} \epsilon_{ij} \epsilon_{kj} = \delta_{ik} \cdot 2^n$, we have

$$\begin{aligned} P_n(r_i^n) &= P_n \left(\frac{1}{2^{n/p}} \sum_{j=1}^{2^n} \epsilon_{ij} e_j^n \right) = \frac{1}{2^{n/p}} \sum_{j=1}^{2^n} \epsilon_{ij} P_n(e_j^n) \\ &= \frac{1}{2^{n/p}} \sum_{j=1}^{2^n} \epsilon_{ij} \left(\frac{1}{2^{n/q}} \sum_{k=1}^n \epsilon_{kj} r_k^n \right) = \frac{1}{2^n} \sum_{j=1}^{2^n} \sum_{k=1}^n \epsilon_{ij} \epsilon_{kj} r_k^n \\ &= \frac{1}{2^n} \sum_{k=1}^n \left(\sum_{j=1}^{2^n} \epsilon_{ij} \epsilon_{kj} \right) r_k^n = \frac{1}{2^n} \sum_{k=1}^n \delta_{ik} \cdot 2^n \cdot r_k^n \\ &= r_i^n. \end{aligned}$$

So we get

$$\begin{aligned} P_n^2(e_k^n) &= \frac{1}{2^{n/q}} \sum_{i=1}^n \epsilon_{ik} P_n(r_i^n) \\ &= \frac{1}{2^{n/q}} \sum_{i=1}^n \epsilon_{ik} r_i^n \\ &= P_n(e_k^n). \end{aligned}$$

So P_n is a projection from $\ell_p^{2^n}$ onto W_n and $P_n(r_i^n) = r_i^n$.

To complete the proof, we need the following:

LEMMA 5.5. *The projections P_n are uniformly bounded in norm.*

This lemma can be deduced implicitly from standard results in the literature (c.f. [DJT, R, FHHMPZ]). However we have not found it stated explicitly in any references. Thus for self completeness we include the following proof which was kindly shown to us by P. Casazza. It is short and self-contained but doesn't give the best uniform bound.

In order to prove Lemma 5.5, we need the following results from Lindenstrauss and Tzafriri:

LEMMA 5.6. *There are constants A_p, B_p so that if $\{r_i\}_{i=1}^m$ are the Rademacher vectors in ℓ_p^n , then for all scalars $\{a_i\}_{i=1}^m$, we have*

$$A_p \left(\sum_{i=1}^m |a_i|^2 \right)^{1/2} \leq \left\| \sum_{i=1}^m a_i r_i \right\|_{\ell_p^n} \leq B_p \left(\sum_{i=1}^m |a_i|^2 \right)^{1/2}.$$

LEMMA 5.7. *If $\{r_i\}_{i=1}^m$ are the Rademacher vectors in ℓ_p^n , then for all $x \in \ell_q^n$,*

$$\left(\sum_{i=1}^m |r_i(x)|^2 \right)^{1/2} \leq B_p \|x\|.$$

PROOF. Give $x \in \ell_q^n$, choose $\{a_i\}_{i=1}^m$ so that $\sum_{i=1}^m |a_i|^2 = 1$ and $(\sum_{i=1}^m |r_i(x)|^2)^{1/2} = \sum_{i=1}^m a_i r_i(x)$. Then we get

$$\begin{aligned} \left(\sum_{i=1}^m |r_i(x)|^2 \right)^{1/2} &= \sum_{i=1}^m a_i r_i(x) = \left(\sum_{i=1}^m a_i r_i \right)(x) \\ &\leq \left\| \sum_{i=1}^m a_i r_i \right\| \|x\| \leq B_p \left(\sum_{i=1}^m |a_i|^2 \right)^{1/2} \|x\| = B_p \|x\|. \end{aligned}$$

□

Proof of Lemma 5.5: Let $\{r_i\}_{i=1}^n$ (respectively, $\{r_i^*\}_{i=1}^n$) be the Rademachers in $\ell_p^{2^n}$ (respectively, $\ell_q^{2^n}$) with $1/p + 1/q = 1$. Then

$$r_i^*(r_j) = \delta_{i,j}.$$

Now we check the norm of P_n . Now, applying our Lemma 5.7 twice,

$$\begin{aligned} \|P_n(x)\| &= \sup_{\|f\|_{\ell_q^{2^n}}=1} |f(P_n(x))| \\ &= \sup_{\|f\|_{\ell_q^{2^n}}=1} \left| \sum_{i=1}^m r_i^*(x) f(r_i) \right| \\ &\leq \sup_{\|f\|_{\ell_q^{2^n}}=1} \left(\sum_{i=1}^m |r_i^*(x)|^2 \right)^{1/2} \left(\sum_{i=1}^m |f(r_i)|^2 \right)^{1/2} \\ &\leq \sup_{\|f\|_{\ell_q^{2^n}}=1} B_q \|x\| B_p \|f\| \\ &= B_p B_q \|x\|. \end{aligned}$$

We complete the proof of Lemma 5.5.

Now we resume the proof of Theorem 5.4: Let Z and W be the Banach spaces defined by

$$Z = \sum_{n=1}^{\infty} \oplus_2 \ell_p^{2^n} = \ell_p^{2^1} \oplus_2 \cdots \oplus_2 \ell_p^{2^n} \oplus_2 \cdots$$

and

$$W = \sum_{n=1}^{\infty} \oplus_2 W_n = W_1 \oplus_2 \cdots \oplus_2 W_n \oplus_2 \cdots,$$

and let P be the projection from Z onto W defined by

$$P = \sum_{n=1}^{\infty} \oplus_2 P_n = P_1 \oplus_2 \cdots \oplus_2 P_n \oplus_2 \cdots.$$

Since $\{e_i^n\}_{i=1}^{2^n}$ is the unconditional unit basis of $\ell_p^{2^n}$, we obtain that

$$\begin{aligned} e_i^1 \oplus_2 0 \oplus_2 \cdots \oplus_2 0 \oplus_2 \cdots, & \quad i = 1, 2 \\ 0 \oplus_2 e_i^2 \oplus_2 \cdots \oplus_2 0 \oplus_2 \cdots, & \quad i = 1, 2, 3, 4 \\ \dots\dots\dots \\ 0 \oplus_2 \cdots \oplus_2 e_i^n \oplus_2 0 \oplus_2 \cdots, & \quad i = 1, 2, \dots, 2^n \\ \dots\dots\dots \end{aligned}$$

is an uncondition basis of Z , and

$$\begin{aligned} (e_i^1)^* \oplus_2 0 \oplus_2 \cdots \oplus_2 0 \oplus_2 \cdots, & \quad i = 1, 2 \\ 0 \oplus_2 (e_i^2)^* \oplus_2 \cdots \oplus_2 0 \oplus_2 \cdots, & \quad i = 1, 2, 3, 4 \\ \dots\dots\dots \\ 0 \oplus_2 \cdots \oplus_2 (e_i^n)^* \oplus_2 0 \oplus_2 \cdots, & \quad i = 1, 2, \dots, 2^n \\ \dots\dots\dots \end{aligned}$$

is the dual of this basis. Thus

$$\left\{ P(0 \oplus_2 \cdots \oplus_2 e_i^n \oplus_2 0 \oplus_2 \cdots), P^*(0 \oplus_2 \cdots \oplus_2 (e_i^n)^* \oplus_2 0 \oplus_2 \cdots) \right\}_{i=1, \dots, 2^n, n=1, 2, \dots}$$

is a framing of W , where

$$P^* = \sum_{n=1}^{\infty} \oplus_2 P_n^* = P_1^* \oplus_2 \cdots \oplus_2 P_n^* \oplus_2 \cdots,$$

and

$$P^*(0 \oplus_2 \cdots \oplus_2 (e_i^n)^* \oplus_2 0 \oplus_2 \cdots) = 0 \oplus_2 \cdots \oplus_2 P^*(e_i^n)^* \oplus_2 0 \oplus_2 \cdots.$$

Since each

$$U_n : W_n \rightarrow \mathcal{H}_n, \quad U_n x = x$$

is an isomorphic operator, it follows that

$$U = \sum_{n=1}^{\infty} \oplus_2 U_n = U_1 \oplus_2 \cdots \oplus_2 U_n \oplus_2 \cdots$$

is an isomorphic operator. This implies that

$$\left\{ U(0 \oplus_2 \cdots \oplus_2 P_n(e_i^n) \oplus_2 0 \oplus_2 \cdots), (U^{-1})^*(0 \oplus_2 \cdots \oplus_2 P_n^*(e_i^n)^* \oplus_2 0 \oplus_2 \cdots) \right\}_{i=1, \dots, 2^n, n=1, 2, \dots}$$

is a framing of the Hilbert space \mathcal{H} , where

$$\mathcal{H} = \sum_{n=1}^{\infty} \oplus_2 \mathcal{H}_n = \mathcal{H}_1 \oplus_2 \cdots \oplus_2 \mathcal{H}_n \oplus_2 \cdots .$$

Now we prove that this framing has a Hilbert space dilation.

Take $\alpha_i^n = 2^{n(1/q-1/2)}$, $i = 1, \dots, 2^n$, $n = 1, 2, \dots$. For any $h = h_1 \oplus_2 \cdots \oplus_2 h_n \oplus_2 \cdots \in \mathcal{H}$, where $h_n = \sum_{k=1}^n a_k^n r_k^n$, $n = 1, 2, \dots$,

$$\begin{aligned} & \langle h, \alpha_i^n U(0 \oplus_2 \cdots \oplus_2 P_n(e_i^n) \oplus_2 0 \oplus_2 \cdots) \rangle_{\mathcal{H}} \\ &= 2^{n(1/q-1/2)} \langle h_1 \oplus_2 \cdots \oplus_2 h_n \oplus_2 \cdots, 0 \oplus_2 \cdots \oplus_2 U_n(P_n(e_i^n)) \oplus_2 0 \oplus_2 \cdots \rangle_{\mathcal{H}} \\ &= 2^{n(1/q-1/2)} \langle h_n, U_n(P_n(e_i^n)) \rangle_{\mathcal{H}_n} \\ &= 2^{n(1/q-1/2)} \langle h_n, P_n(e_i^n) \rangle_{\mathcal{H}_n} . \end{aligned}$$

From $P_n(e_i^n) = \frac{1}{2^{n/q}} \sum_{j=1}^n \epsilon_{ji} r_j^n$, we get that

$$\langle h_n, P_n(e_i^n) \rangle_{\mathcal{H}_n} = \left\langle \sum_{k=1}^n a_k^n r_k^n, \frac{1}{2^{n/q}} \sum_{j=1}^n \epsilon_{ji} r_j^n \right\rangle_{\mathcal{H}_n} = \frac{1}{2^{n/q}} \sum_{k=1}^n a_k^n \epsilon_{ki} .$$

Thus

$$\begin{aligned} & \langle h, \alpha_i^n U(0 \oplus_2 \cdots \oplus_2 P_n(e_i^n) \oplus_2 0 \oplus_2 \cdots) \rangle_{\mathcal{H}} \\ &= 2^{n(1/q-1/2)} \cdot \frac{1}{2^{n/q}} \sum_{k=1}^n a_k^n \epsilon_{ki} \\ (5.3) \quad &= 2^{-n/2} \sum_{k=1}^n a_k^n \epsilon_{ki} . \end{aligned}$$

On the other hand,

$$\begin{aligned} & \left\langle h, \frac{1}{\alpha_i^n} (U^{-1})^* (0 \oplus_2 \cdots \oplus_2 P_n^*(e_i^n)^* \oplus_2 0 \oplus_2 \cdots) \right\rangle_{\mathcal{H}} \\ &= 2^{n(1/2-1/q)} \langle h, 0 \oplus_2 \cdots \oplus_2 (U_n^{-1})^* (P_n^*(e_i^n)^*) \oplus_2 0 \oplus_2 \cdots \rangle_{\mathcal{H}} \\ &= 2^{n(1/2-1/q)} \langle h_n, (U_n^{-1})^* (P_n^*(e_i^n)^*) \rangle_{\mathcal{H}_n} \\ &= 2^{n(1/2-1/q)} P_n^*(e_i^n)^* (U_n^{-1} h_n) \\ &= 2^{n(1/2-1/q)} P_n^*(e_i^n)^* (h_n) \\ &= 2^{n(1/2-1/q)} (e_i^n)^* (P_n(h_n)) \\ &= 2^{n(1/2-1/q)} (e_i^n)^* (h_n) \\ &= 2^{n(1/2-1/q)} (e_i^n)^* \left(\sum_{k=1}^n a_k^n r_k^n \right) \\ &= 2^{n(1/2-1/q)} \sum_{k=1}^n a_k^n (e_i^n)^* (r_k^n) . \end{aligned}$$

Since

$$(e_i^n)^* (r_k^n) = (e_i^n)^* \left(\frac{1}{2^{n/p}} \sum_{j=1}^{2^n} \epsilon_{kj} e_j^n \right) = \frac{1}{2^{n/p}} \sum_{j=1}^{2^n} \epsilon_{kj} (e_i^n)^* (e_j^n) = \frac{1}{2^{n/p}} \epsilon_{ki} ,$$

we have

$$\begin{aligned}
 & \left\langle h, \frac{1}{\alpha_i^n} (U^{-1})^* (0 \oplus_2 \cdots \oplus_2 P_n^*(e_i^n)^* \oplus_2 0 \oplus_2 \cdots) \right\rangle_{\mathcal{H}} \\
 &= 2^{n(1/2-1/q)} \sum_{k=1}^n a_k^n \frac{1}{2^{n/p}} \epsilon_{ki} \\
 &= 2^{n(1/2-1/q-1/p)} \sum_{k=1}^n a_k^n \epsilon_{ki} \\
 (5.4) \quad &= 2^{-n/2} \sum_{k=1}^n a_k^n \epsilon_{ki}.
 \end{aligned}$$

From (5.3) and (5.3), we know that

$$\alpha_i^n U (0 \oplus_2 \cdots \oplus_2 P_n(e_i^n) \oplus_2 0 \oplus_2 \cdots) = \frac{1}{\alpha_i^n} (U^{-1})^* (0 \oplus_2 \cdots \oplus_2 P_n^*(e_i^n)^* \oplus_2 0 \oplus_2 \cdots).$$

Now we show that

$$\left\{ \alpha_i^n U (0 \oplus_2 \cdots \oplus_2 P_n(e_i^n) \oplus_2 0 \oplus_2 \cdots) \right\}_{i=1, \dots, 2^n, n=1, 2, \dots}$$

is the Parseval frame of \mathcal{H} .

For any $h = h_1 \oplus_2 \cdots \oplus_2 h_n \oplus_2 \cdots \in \mathcal{H}$, where $h_n = \sum_{k=1}^n a_k^n r_k^n$, $n = 1, 2, \dots$, we have

$$\|h\|^2 = \sum_{n=1}^{\infty} \|h_n\|^2 = \sum_{n=1}^{\infty} \sum_{k=1}^n |a_k^n|^2.$$

From $\sum_{j=1}^{2^n} \epsilon_{ij} \epsilon_{kj} = \delta_{ik} \cdot 2^n$, for any $\{a_k^n\}_{k=1, \dots, n}$, we have

$$\sum_{i=1}^{2^n} \frac{1}{2^n} \left| \sum_{k=1}^n a_k^n \epsilon_{ki} \right|^2 = \sum_{k=1}^n |a_k^n|^2.$$

Hence we get

$$\begin{aligned}
 & \sum_{n=1}^{\infty} \sum_{i=1}^{2^n} |\langle h, \alpha_i^n U (0 \oplus_2 \cdots \oplus_2 P_n(e_i^n) \oplus_2 0 \oplus_2 \cdots) \rangle_{\mathcal{H}}|^2 \\
 &= \sum_{n=1}^{\infty} \sum_{i=1}^{2^n} \left| \frac{1}{2^{n/2}} \sum_{k=1}^n a_k^n \epsilon_{ki} \right|^2 \\
 &= \sum_{n=1}^{\infty} \sum_{i=1}^{2^n} \frac{1}{2^n} \left| \sum_{k=1}^n a_k^n \epsilon_{ki} \right|^2 \\
 &= \sum_{n=1}^{\infty} \sum_{k=1}^n |a_k^n|^2 \\
 &= \|h\|^2.
 \end{aligned}$$

Therefore

$$\left\{ \alpha_i^n U (0 \oplus_2 \cdots \oplus_2 P_n(e_i^n) \oplus_2 0 \oplus_2 \cdots) \right\}_{i=1, \dots, 2^n, n=1, 2, \dots}$$

is the Parseval frame of \mathcal{H} . Finally by Theorem 3.8, we obtain that

$\left\{ U \left(0 \oplus_2 \cdots \oplus_2 P_n(e_i^n) \oplus_2 0 \oplus_2 \cdots \right), (U^{-1})^* \left(0 \oplus_2 \cdots \oplus_2 P_n^*(e_i^n)^* \oplus_2 0 \oplus_2 \cdots \right) \right\}_{i=1, \dots, 2^n, n=1, 2, \dots}$
 has a Hilbert space dilation. This completes the proof of Theorem 5.4. \square

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